

Analysis of Lepton Flavor Violating $\tau^\pm \rightarrow \mu^\pm \mu^\pm \mu^\mp$ Decays

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Abstract

Supposing only Lorentz and the gauge invariances of the Lagrangian, we derive energy and angular distributions for $\tau^\pm \rightarrow \mu^\pm \mu^\pm \mu^\mp$ lepton flavor violating decay process. Using these results, we discuss methods to determine the parameters associated with the lepton flavor violating interactions.

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1 Introduction

Lepton flavor violating (LFV) decays, for example, $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$, μ - e conversion and $\tau \rightarrow 3\mu$ are being studied extensively by experimentists [1]-[4], and by theorists [5]-[8]. Especially, a large number of τ decay events are collected in B Factories and we hope that LFV decay mode of τ may be found in the near future. In that case, the Super B Factory [9] might give us the large number of $\tau \rightarrow 3\mu$ events. We might also hope to observe the energy and polarization distributions.

In $\tau \rightarrow 3\mu$ decay, we can measure all the energies and directions of the final state muons. For definiteness, let's say that we want to investigate the τ^+ decay. B Factories generate back to back $\tau^+\tau^-$ pairs. As mentioned in Section 2, the polarization of τ^+ can be observed statistically by correlation of momenta of decay products of both τ^+ and τ^- [10]. For $\mu^+ \rightarrow e^+e^+e^-$ process, the differential branching ratio which is a function of energies of 2 positrons in the final state and the polarization of μ^+ in initial state is already derived in Ref. [5]. The structure of weak interaction which causes the $\mu \rightarrow e\nu\bar{\nu}$ decay was investigated by Michel [11]. He introduced Michel parameters which proved to be very useful. We follow the similar strategies and formulate the method with which we can probe the structure of LFV interactions starting from a general Lagrangian.

The general Lagrangian for τ^+ decaying to $\mu^+\mu^+\mu^-$ supposing Lorentz invariance and gauge invariance is written as:

$$\mathcal{L} = \mathcal{L}_{FF} + \mathcal{L}_\gamma, \quad (1)$$

$$\begin{aligned} \mathcal{L}_{FF} = -2\sqrt{2}G_F \Big\{ & g_1(\bar{\tau}_R\mu_L)(\bar{\mu}_R\mu_L) + g_2(\bar{\tau}_L\mu_R)(\bar{\mu}_L\mu_R) \\ & + g_3(\bar{\tau}_R\gamma_\alpha\mu_R)(\bar{\mu}_R\gamma^\alpha\mu_R) + g_4(\bar{\tau}_L\gamma_\alpha\mu_L)(\bar{\mu}_L\gamma^\alpha\mu_L) \\ & + g_5(\bar{\tau}_R\gamma_\alpha\mu_R)(\bar{\mu}_L\gamma^\alpha\mu_L) + g_6(\bar{\tau}_L\gamma_\alpha\mu_L)(\bar{\mu}_R\gamma^\alpha\mu_R) \Big\}, \end{aligned} \quad (2)$$

$$\begin{aligned} \mathcal{L}_\gamma = -2\sqrt{2}G_F m_\tau \Big\{ & A_R\bar{\tau}_R\sigma^{\alpha\beta}\mu_L F_{\alpha\beta} + A_L\bar{\tau}_L\sigma^{\alpha\beta}\mu_R F_{\alpha\beta} \Big\} \\ & + \bar{\mu}(iD^\alpha\gamma_\alpha - m_\mu)\mu - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}, \end{aligned} \quad (3)$$

where m_μ and m_τ are the masses of the μ^\pm and τ^\pm , respectively, G_F is the Fermi constant. $\{\bar{\tau}_L, \bar{\mu}_L, \mu_R\}$ and $\{\bar{\tau}_R, \bar{\mu}_R, \mu_L\}$ are the Dirac spinors with the helicity operators, $(1 \pm \gamma_5)/2$, respectively. For example, $\bar{\tau}_R = \bar{\tau}(1 - \gamma_5)/2$ and $\mu_R = (1 + \gamma_5)\mu/2$. $\sigma^{\alpha\beta} = \frac{i}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$, $D^\alpha = \partial^\alpha + ieA^\alpha$, $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, A^α is the photon field and $e = -|e|$ is the electron charge. \mathcal{L}_γ represents $\tau \rightarrow \mu\gamma$ interaction as well as the terms present in

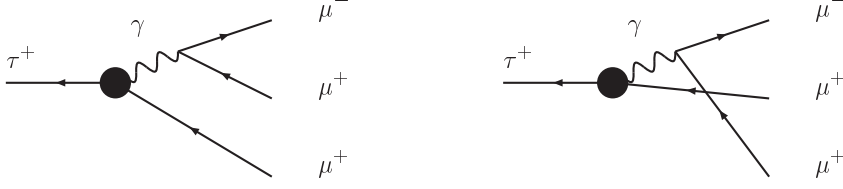


Figure 1: These are the diagrams produced from \mathcal{L}_γ .

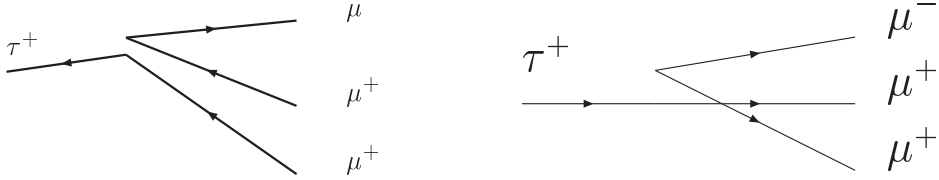


Figure 2: These are the diagrams produced from \mathcal{L}_{FF} .

ordinary QED. These terms generate diagrams in Fig. 1. So, A_L and A_R are the coefficients of interactions in which the intermediate photon has the left polarization, and the right polarization, respectively. The coefficients $g_1 \sim g_6$ in \mathcal{L}_{FF} are the coefficients of various 4-Fermi type interactions. These terms generate diagrams in Fig. 2. Using the Fierz transformation, it is shown in the Appendix A that Eq. (1) is the general Lagrangian.

For convenience, we summarize our major results, here assuming that enough events are collected.

1. We have shown that the absolute values of the coefficients $|g_1|^2/16 + |g_3|^2 + |g_2|^2/16 + |g_4|^2$, $|g_5|^2 + |g_6|^2$, $|A_R|^2 + |A_L|^2$, $Re[g_3 A_L^*] + Re[g_4 A_R^*]$ and $Re[g_5 A_L^*] + Re[g_6 A_R^*]$ can be obtained by measuring the energy distributions of decay products of τ^+ .
2. We have shown that the absolute values of the coefficients $|g_1|^2/16 + |g_3|^2$, $|g_2|^2/16 + |g_4|^2$, $|g_5|$, $|g_6|$, $|A_R|$, $|A_L|$, $Re[g_3 A_L^*]$, $Re[g_4 A_R^*]$, $Re[g_5 A_L^*]$, $Re[g_6 A_R^*]$, $Im[g_3 A_L^*] + Im[g_4 A_R^*]$ and $Im[g_5 A_L^*] + Im[g_6 A_R^*]$ can be obtained by measuring the momentum distribution of decay products of τ^+ in addition to the energy distributions of decay products of τ^+ .
3. In some suitable cases, we can determine $|g_1|$, $|g_2|$, $|g_3|$ and $|g_4|$, independently, and phases $\arg[g_4 A_R^*]$, $\arg[g_3 A_L^*]$, $\arg[g_6 A_R^*]$ and $\arg[g_5 A_L^*]$ as shown in section 9.
4. It is interesting to note that there is a possibility that some information on $\tau \rightarrow \mu\gamma$ could be obtained before $\tau \rightarrow \mu\gamma$ decay is measured. For example, $Re[g_4 A_R^*]$ and $Re[g_3 A_L^*]$ arise from the interference terms between the four Fermi interaction amplitudes and the $\tau \rightarrow \mu\gamma$ decay amplitudes. The decay rate for $\tau \rightarrow \mu\gamma$ is quadratic in $|A_R|$ or $|A_L|$

while the observables given in $Re[g_4 A_R^*]$ and $Re[g_3 A_L^*]$ are linear in $|A_R|$ and $|A_L|$. So the interference effect may be seen before the decay rate for $\tau \rightarrow \mu\gamma$ is seen.

This paper is organized as follows. Section 2 gives the differential branching ratio of $\tau^+ \rightarrow \mu^+ \mu^+ \mu^-$ decay. Also, a general formula which yields information on the current structure for $\tau^+ \rightarrow \mu^+ \mu^+ \mu^-$ decay is derived. In section 3, we pick energetic one of the two μ^+ in the final state and analyze its energy dependence. In section 4, we discuss physics implications of our results using only the results of section 3. In section 5, we give the energy dependence of μ^- . In section 6, we discuss physics implications of our results using only the results of section 5. In section 7, we discuss physics implications of our results using the results of sections 4 and 6. In section 8, we give τ^+ polarization dependence of the branching ratio in addition to the energetic μ^+ energy dependence and analyze the results. In section 9, we discuss physics implications of our results using the results derived until previous section. Section 10 contains the concluding remarks.

2 General Formula

In this section, we derive the differential branching ratio for $\tau^+ \rightarrow \mu^+ \mu^+ \mu^-$ decay including τ^+ polarization, and the general formula for the observables which are relevant for the actual experimental situation.

The final state contains two μ^+ mesons. The one with higher energy is denoted as μ_1 . The other is denoted as μ_2 . μ^- is denoted as μ_3 .

In the real experiment, the processes we want to detect are

$$\begin{aligned} e^+ e^- \rightarrow \tau^+(s^+) \tau^-(s^-) \rightarrow \nu_\tau + a + \text{anything} \\ \quad \quad \quad \hookrightarrow \mu_1 \mu_2 \mu_3 \end{aligned} \quad (4)$$

where a is a particle which has the charge -1 . So we must calculate the differential cross section for these processes.

Here, we use the center of mass frame of $e^+ e^-$ initial state, which we call frame 1. In the rest frame of τ^+ , which we name frame 3, the momenta of μ_1 , μ_2 , μ_3 and τ^+ are denoted as $p_1 = (E_1, \mathbf{p}_1)$, $p_2 = (E_2, \mathbf{p}_2)$, $p_3 = (E_3, \mathbf{p}_3)$ and $p_\tau = (m_\tau, \mathbf{0})$, and finally the polarization of τ^+ is denoted as s^+ . s^- denotes the polarization of τ^- in the rest frame of τ^- . Here, $s^+ \cdot p_\tau = s^- \cdot k_\tau = 0$ and $(s^\pm)^2 = -1$ where k_τ is the momentum of τ^- in the τ^- rest frame which we call frame 2.

For the definiteness, we set the relations between frame 1, 2, 3, as follows and they are depicted in Fig. 3.

Frame 1 is the center of mass frame of $e^+ e^-$. Defining the τ^+ momentum as \mathbf{p}'_τ and the initial state positron momentum as \mathbf{p}'_{e^+} , we set the z

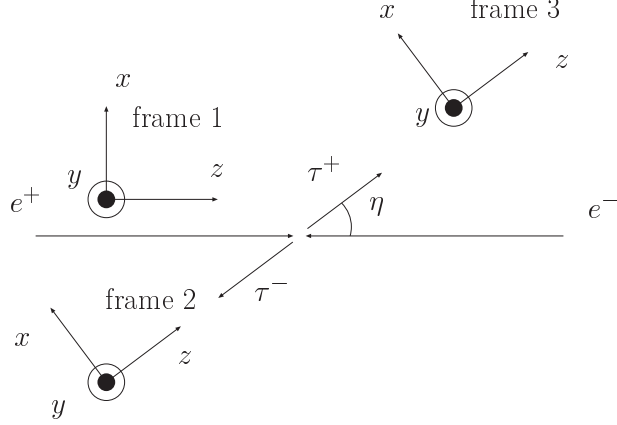


Figure 3: The relation between frame 1, frame 2 and frame 3. η is the angle between \mathbf{p}'_{e^+} and \mathbf{p}'_{τ^-} .

direction in this frame as the direction of \mathbf{p}'_{e^+} , y direction is the same as $\mathbf{p}'_{e^+} \times \mathbf{p}'_{\tau^-}$ and the x direction as $(\mathbf{p}'_{e^+} \times \mathbf{p}'_{\tau^-}) \times \mathbf{p}'_{e^+}$.

Frame 2 is the τ^- rest frame. We set the z direction in this frame as the direction of \mathbf{p}'_{τ^-} . The y direction is the same as that of the frame 1. The x direction is defined by $(\mathbf{p}'_{e^+} \times \mathbf{p}'_{\tau^-}) \times \mathbf{p}'_{\tau^-}$.

Frame 3 is the τ^+ rest frame, We set the directions in this frame are the same as that of the frame 2.

There are four ingredients that are used to compute the differential cross section for the process shown in Eq. (4).

1. The narrow width approximation where we approximate

$$e^+e^- \rightarrow \tau^+(s^+) \tau^-(s^-) \rightarrow \nu_{\tau} + a + \text{anything} \\ \quad \quad \quad \hookrightarrow \mu_1 \mu_2 \mu_3$$

is described in appendix B.3.

2. The differential cross section for $e^+e^- \rightarrow \gamma^* \rightarrow \tau^+(s^+)\tau^-(s^-)$ is presented in appendix B.1.

3. The differential branching ratio for $\tau^-(s^-) \rightarrow \nu_\tau + a + \text{anything}$ is presented in appendix B.2.
4. The differential branching ratio for $\tau^+(s^+) \rightarrow \mu_1\mu_2\mu_3$ is given below.

The differential cross section for this process $d\sigma$ is written as [12]

$$\begin{aligned}
& \frac{d\sigma}{d\Omega dx_1 dx_2 d\Omega_\tau d\psi d^3k_a} \\
&= \sum_{s^+, s^-} \sum_{\pm s^+, \pm s^-} \frac{d\sigma(e^+e^- \rightarrow \tau^+(s^+)\tau^-(s^-))}{d\Omega} \\
& \quad \times \frac{dBr(\tau^-(s^-) \rightarrow \nu_\tau + a + \text{anything})}{d^3k_a} \\
& \quad \times \frac{dBr(\tau^+(s^+) \rightarrow \mu_1\mu_2\mu_3)}{dx_1 dx_2 d\Omega_\tau d\psi}.
\end{aligned} \tag{5}$$

\mathbf{S} implies sum over polarizations. k_a is the momentum of the particle a in τ^- rest frame. The definitions of Ω_τ and ψ are written in next paragraph.

We have computed the branching ratio for $\tau^+(s^+) \rightarrow \mu_1\mu_2\mu_3$ including m_μ dependences. However, for now, we confine our discussion where we can approximate $m_\mu/m_\tau = 0$. The result is

$$\begin{aligned}
& \frac{dBr(\tau^+(s^+) \rightarrow \mu_1\mu_2\mu_3)}{dx_1 dx_2 d\Omega_\tau d\psi} \\
&= \frac{3}{2\pi^2} Br(\tau \rightarrow \mu\nu\bar{\nu}) \left[G_0(x_1, x_2) + \sum_i \mathbf{s}^+ \cdot \mathbf{P}_i G_i^s(x_1, x_2) \right],
\end{aligned} \tag{6}$$

where the definitions of $G_0(x_1, x_2)$ and $G_i^s(x_1, x_2)$ are written in Appendix C. $Br(\tau \rightarrow \mu\nu\bar{\nu})$ is the branching ratio of $\tau^+ \rightarrow \mu^+\nu_\mu\bar{\nu}_\tau$ decay.

$$x_1 = \frac{2E_1}{m_\tau}, \quad x_2 = \frac{2E_2}{m_\tau}, \quad x_3 = \frac{2E_3}{m_\tau}. \tag{7}$$

Note that x_1, x_2 and x_3 take values between 0 and 1, and $x_1 + x_2 + x_3 = 2$. Ω_τ is the solid angle defined in Fig. 4 and ψ is the angle between the \mathbf{p}_2 - \mathbf{p}_3 plane and z - \mathbf{p}_3 plane as defined in Fig. 5. $\mathbf{P}_i = \{\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2\}$ where $\hat{\mathbf{p}}_1 = \mathbf{p}_1/|\mathbf{p}_1|$ and $\hat{\mathbf{p}}_2 = \mathbf{p}_2/|\mathbf{p}_2|$.

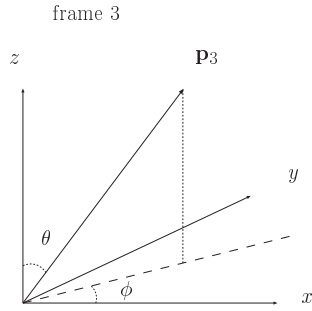


Figure 4: Definition of θ and ϕ in frame 3

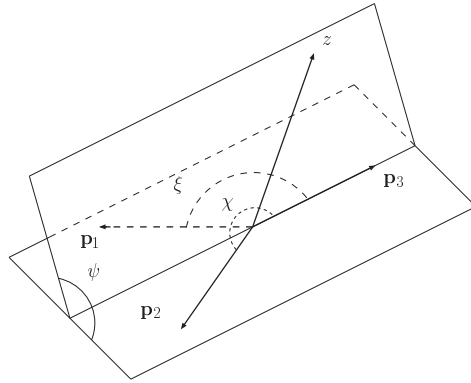


Figure 5: ψ is the angle between \mathbf{p}_2 - \mathbf{p}_3 plane and \mathbf{p}_3 - z plane. ξ is the angle between \mathbf{p}_3 and \mathbf{p}_1 . χ is the angle between \mathbf{p}_3 and \mathbf{p}_2 . $0 \leq \xi \leq \pi$. $\pi \leq \chi \leq 2\pi$.

Substituting the concrete representations, Eq. (5) becomes

$$\begin{aligned}
& \frac{d\sigma}{d\Omega \, dx_1 dx_2 d\Omega_\tau d\psi \, d^3k_a} \\
&= \sum_{s^+, s^-} \sum_{\pm s^+, \pm s^+} \frac{\alpha^2 \beta}{4q^2} \left[(1 + \cos^2 \eta + \frac{\sin^2 \eta}{\gamma^2}) + (1 + \cos^2 \eta - \frac{\sin^2 \eta}{\gamma^2}) s_z^+ s_z^- \right. \\
&\quad \left. - \beta^2 \sin^2 \eta \, s_y^+ s_y^- + (1 + \frac{1}{\gamma^2}) \sin^2 \eta \, s_x^+ s_x^- - \frac{\sin 2\eta}{\gamma} (s_z^+ s_x^- + s_x^+ s_z^-) \right] \\
&\quad \times \frac{3}{2\pi^2} Br(\tau \rightarrow \mu \nu \bar{\nu}) \left[G_0(x_1, x_2) + \sum_i s^+ \cdot \mathbf{P}_i G_i^s(x_1, x_2) \right] \\
&\quad \times Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{2}{\pi m_\tau^3 \lambda_a} \left[G_1^a(y_a) - \mathbf{s}^- \cdot \hat{\mathbf{k}}_a G_2^a(y_a) \right].
\end{aligned} \tag{8}$$

Carrying out the summation for $\pm s^+$ and $\pm s^-$ using $s_i^\pm s_j^\pm = \delta_{ij}$, Eq. (8) becomes

$$\begin{aligned}
& \frac{d\sigma}{d\Omega \, dx_1 dx_2 d\Omega_\tau d\psi \, d^3k_a} \\
&= 4Br(\tau \rightarrow \mu \nu \bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{3\alpha^2 \beta}{4\pi^3 q^2 m_\tau^3 \lambda_a} \\
&\quad \times \left[G_0(x_1, x_2) G_1^a(y_a) (1 + \cos^2 \eta + \frac{\sin^2 \eta}{\gamma^2}) \right. \\
&\quad \left. - \sum_i G_i^s(x_1, x_2) G_2^a(y_a) \left\{ (1 + \cos^2 \eta - \frac{\sin^2 \eta}{\gamma^2}) \hat{k}_{az} P_{iz} - \beta^2 \sin^2 \eta \, \hat{k}_{ay} P_{iy} \right. \right. \\
&\quad \left. \left. + (1 + \frac{1}{\gamma^2}) \sin^2 \eta \, \hat{k}_{ax} P_{ix} - \frac{\sin 2\eta}{\gamma} (\hat{k}_{ax} P_{iz} + \hat{k}_{az} P_{ix}) \right\} \right].
\end{aligned} \tag{9}$$

This expression is the general formula of the process (4) as long as we stay sufficiently away from singularity at $m_\mu = 0$. In following sections, we'll start analyzing from this expression.

3 $G_0(x_1)$: Energy Dependence of μ_1

In this section, we derive the formulae convenient for investigating the structure of the LFV four Fermi interactions, using the μ_1 energy dependence of the differential branching ratio for $\tau \rightarrow 3\mu$ decay mode.

Our first priority is to discuss the observable which is easier to detect and analyze. So here, we integrate the polarization dependence as follows.

First, we integrate over $d\Omega d\Omega_\tau d\psi$,

$$\begin{aligned} & \frac{d\sigma}{dx_1 dx_2 d^3 k_a} \\ &= Br(\tau \rightarrow \mu \nu \bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{64\alpha^2 \beta}{q^2 m_\tau^3 \lambda_a} G_0(x_1, x_2) G_1^a(y_a) (2 + \frac{1}{\gamma^2}). \end{aligned} \quad (10)$$

Next, we integrate over $d^3 k_a$,

$$\begin{aligned} & \frac{d\sigma}{dx_1 dx_2} \\ &= \frac{32\pi\alpha^2 \beta}{q^2} (2 + \frac{1}{\gamma^2}) Br(\tau \rightarrow \mu \nu \bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \\ & \quad \times G_0(x_1, x_2). \end{aligned} \quad (11)$$

Eq. (11) allow us to obtain $G_0(x_1, x_2)$. It has the arguments x_1 and x_2 . Even expression, however, is pretty complicated for a discussion here. We thus discuss only the x_1 dependence integrating over x_2 . For the physical region, we found that the effect of neglecting muon masses in the differential branching ratio for the $\tau \rightarrow 3\mu$ decay introduces an error of $\mathcal{O}(2m_\mu/m_\tau)$.

Now we define

$$\begin{aligned} a_+ &= (\frac{|g_1|^2}{16} + |g_3|^2) + (\frac{|g_2|^2}{16} + |g_4|^2) \\ b_+ &= |g_5|^2 + |g_6|^2 \\ c_+ &= |eA_R|^2 + |eA_L|^2 \\ d_+ &= -(Re[g_3 eA_L^*] + Re[g_4 eA_R^*]) \\ e_+ &= -(Re[g_6 eA_R^*] + Re[g_5 eA_L^*]). \end{aligned} \quad (12)$$

Then, G_0 integrated over x_2 is

$$\begin{aligned} G_0(x_1) &= \int_{1-x_1}^{x_1} dx_2 G_0(x_1, x_2) \\ &= \frac{1}{3} J_1(2x_1 - 1) \left\{ 6(1 - x_1)(2x_1 - 1) \right. \\ & \quad \left. + \rho_{1a}(6x_1 - 5)(2x_1 - 1) \right. \\ & \quad \left. + \frac{8}{3} \rho_{1b}(3x_1 - 2)(x_1 - 1) \right\} \\ & \quad + \frac{4}{3} c_+ \left\{ \frac{2(2x_1 - 1)(x_1^2 - x_1 + 1)}{(1 - x_1)} + 3(2x_1^2 - 2x_1 + 1) \log \left[\frac{x_1}{1 - x_1} \right] \right\}, \end{aligned} \quad (13)$$

where

$$\begin{aligned}
J_1 &= \frac{1}{3}(10a_+ + 7b_+ + 72d_+ + 54e_+) \\
J_1\rho_{1a} &= \frac{1}{2}(4a_+ + b_+ + 48d_+ + 12e_+) \\
J_1\rho_{1b} &= \frac{9}{4}(b_+ + 8e_+).
\end{aligned} \tag{14}$$

We have defined ρ_{1a} and ρ_{1b} to take values between 0 and 1. We now discuss how c_+ , J_1 , ρ_{1a} and ρ_{1b} can be determined from the x_1 dependence of G_0 .

First, to determine c_+ , it is convenient to define the function

$$F_1(x_1) = \frac{3}{8}(1 - x_1)G_0(x_1). \tag{15}$$

By choosing the kinematics such that $x_1 \rightarrow 1$ for the function $F_1(x_1)$, we can obtain c_+ as

$$F_1(x_1)|_{x_1=1} = c_+(2x_1 - 1)(x_1^2 - x_1 + 1)|_{x_1=1} = c_+. \tag{16}$$

Next, we subtract the term containing the coefficient c_+ and define another function,

$$\begin{aligned}
F_2(x_1) &= \frac{G_0(x_1) - (c_+, x_1 \text{ term})}{(2x_1 - 1)} \\
&= \frac{1}{3}J_1 \left\{ 6(1 - x_1)(2x_1 - 1) \right. \\
&\quad \left. + \rho_{1a}(6x_1 - 5)(2x_1 - 1) \right. \\
&\quad \left. + \frac{8}{3}\rho_{1b}(3x_1 - 2)(x_1 - 1) \right\},
\end{aligned} \tag{17}$$

where

$$(c_+, x_1 \text{ term}) = \frac{4}{3}c_+ \left\{ \frac{2(2x_1 - 1)(x_1^2 - x_1 + 1)}{(1 - x_1)} + 3(2x_1^2 - 2x_1 + 1) \log \left[\frac{x_1}{1 - x_1} \right] \right\}. \tag{18}$$

We can then determine the parameters J_1 , ρ_{1a} and ρ_{1b} from the shape of $F_2(x_1)$ using

$$\int_{1/2}^1 F_2(x_1) dx_1 = \frac{1}{12}J_1, \tag{19}$$

$$F_2(x_1)|_{x_1=1} = \frac{1}{3}J_1\rho_{1a} \tag{20}$$

and

$$F_2(x_1)|_{x_1=\frac{1}{2}} = \frac{2}{9}J_1\rho_{1b}, \tag{21}$$

respectively.

4 Physics from $G_0(x_1)$ Distribution of μ_1

The conclusion of previous section is how to determine the parameters, c_+ , J_1 , ρ_{1a} , ρ_{1b} . Using only these parameters, a_+ , b_+ , d_+ and e_+ can't be determined. However, if we take some special cases which are explained bellow, we can restrict the allowed regions of a_+ , b_+ , d_+ and e_+ .

4.1 What happens if one of a_+ , b_+ , c_+ , d_+ , e_+ = 0

If we impose that one of a_+ , b_+ , c_+ , d_+ and e_+ is zero from other experimental results or supposing some specific models, the results are as follows. When $a_+ = 0$, also $d_+ = 0$ and

$$\begin{aligned} J_1 &= \frac{1}{3}(7b_+ + 54e_+) \\ J_1\rho_{1a} &= \frac{1}{2}(b_+ + 12e_+) \\ J_1\rho_{1b} &= \frac{9}{4}(b_+ + 8e_+). \end{aligned} \tag{22}$$

When $b_+ = 0$, also $e_+ = 0$ and

$$\begin{aligned} J_1 &= \frac{1}{3}(10a_+ + 72d_+) \\ J_1\rho_{1a} &= \frac{1}{2}(4a_+ + 48d_+) \\ J_1\rho_{1b} &= 0. \end{aligned} \tag{23}$$

When $c_+ = 0$, also $d_+ = e_+ = 0$ and

$$\begin{aligned} J_1 &= \frac{1}{3}(10a_+ + 7b_+) \\ J_1\rho_{1a} &= \frac{1}{2}(4a_+ + b_+) \\ J_1\rho_{1b} &= \frac{9}{4}b_+. \end{aligned} \tag{24}$$

When $d_+ = 0$,

$$\begin{aligned} J_1 &= \frac{1}{3}(10a_+ + 7b_+ + 54e_+) \\ J_1\rho_{1a} &= \frac{1}{2}(4a_+ + b_+ + 12e_+) \\ J_1\rho_{1b} &= \frac{9}{4}(b_+ + 8e_+). \end{aligned} \tag{25}$$

When $e_+ = 0$,

$$\begin{aligned} J_1 &= \frac{1}{3}(10a_+ + 7b_+ + 72d_+) \\ J_1\rho_{1a} &= \frac{1}{2}(4a_+ + b_+ + 48d_+) \\ J_1\rho_{1b} &= \frac{9}{4}b_+. \end{aligned} \tag{26}$$

If any one of a_+ , b_+ , c_+ , d_+ and e_+ is zero, we can determine a_+ , b_+ , c_+ , d_+ and e_+ from only the x_1 distribution.

4.2 Results if either $a_+ \neq 0$, $b_+ \neq 0$, $d_+ \neq 0$ or $e_+ \neq 0$

Similarly, When only $a_+ \neq 0$,

$$\begin{aligned} J_1 &= \frac{10}{3}a_+ \\ \rho_{1a} &= \frac{3}{5} \\ \rho_{1b} &= 0. \end{aligned} \tag{27}$$

When only $b_+ \neq 0$,

$$\begin{aligned} J_1 &= \frac{7}{3}b_+ \\ \rho_{1a} &= \frac{3}{14} \\ \rho_{1b} &= \frac{27}{28}. \end{aligned} \tag{28}$$

When only a_+ , c_+ , $d_+ \neq 0$ but a_+ is negligible,

$$\begin{aligned} J_1 &= 24d_+ \\ \rho_{1a} &= 1 \\ \rho_{1b} &= 0. \end{aligned} \tag{29}$$

When only b_+ , c_+ , $e_+ \neq 0$ but b_+ is negligible,

$$\begin{aligned} J_1 &= 18e_+ \\ \rho_{1a} &= \frac{1}{3} \\ \rho_{1b} &= 1. \end{aligned} \tag{30}$$

In these case, ρ_{1a} and ρ_{1b} are independent from a_+ , b_+ , d_+ and e_+ .

4.3 What happens if $\tau \rightarrow \mu\gamma$ dominates

If $\tau \rightarrow \mu\gamma$ type interaction which has coefficient c_+ is much larger than other interactions, only c_+ , d_+ and e_+ may be determined since d_+ and e_+ are not quadratic but linier of 4-fermi type interactions. In this case,

$$\begin{aligned} J_1 &= 6(4d_+ + 3e_+) \\ J_1\rho_{1a} &= 6(4d_+ + e_+) \\ J_1\rho_{1b} &= 18e_+. \end{aligned} \tag{31}$$

From Eqs. (234) and (236) in appendix D,

$$\begin{aligned} a_+ &\geq \frac{d_+^2}{c_+} = \frac{1}{c_+} \left(\frac{J_1(1 - \rho_{1b})}{24} \right)^2 \\ b_+ &\geq \frac{e_+^2}{c_+} = \frac{1}{c_+} \left(\frac{J_1\rho_{1b}}{18} \right)^2. \end{aligned} \tag{32}$$

So, we can determine a_+ and b_+ lower limits, though we can't determine a_+ and b_+ , directly.

4.4 Lower limits of c_+ when $\tau \rightarrow \mu\gamma$ is highly suppressed

We give three types of c_+ lower limits. Especially, these are very useful when $\tau \rightarrow \mu\gamma$ is highly suppressed compared with $\tau \rightarrow 3\mu$ and c_+ cannot determine directly.

First, we give the relations,

$$\begin{aligned} \frac{4}{9}J_1\rho_{1b} &= b_+ + 8e_+ \\ 3J_1 - 5J_1\rho_{1a} - 2J_1\rho_{1b} &= -12(4d_+ + e_+) \\ 3J_1 - 3J_1\rho_{1a} - \frac{22}{9}J_1\rho_{1b} &= 4(a_+ - 2e_+) \\ 3J_1 + J_1\rho_{1a} - \frac{10}{3}J_1\rho_{1b} &= 12(a_+ + 8d_+) \\ 6J_1 - 10J_1\rho_{1a} - \frac{8}{3}J_1\rho_{1b} &= 3(b_+ - 32d_+) \\ 3J_1 - 3J_1\rho_{1a} - 2J_1\rho_{1b} &= 4a_+ + b_+ \end{aligned} \tag{33}$$

from Eqs. (14).

Using Eqs. (33), (234) and (236),

$$\begin{aligned}
\frac{1}{12^2 \times 5} \frac{(3J_1 - 5J_1\rho_{1a} - 2J_1\rho_{1b})^2}{3J_1 - 3J_1\rho_{1a} - 2J_1\rho_{1b}} &= \frac{(4d_+ + e_+)^2}{5(4a_+ + b_+)} \\
&\leq \frac{(4d_+ + e_+)^2}{5\left(4\frac{d_+^2}{c_+} + \frac{e_+^2}{c_+}\right)} \\
&= c_+ \frac{(4\frac{d_+}{e_+} + 1)^2}{5\left(4\frac{d_+^2}{e_+^2} + 1\right)} \\
&\leq c_+.
\end{aligned} \tag{34}$$

This becomes equality when $a_+c_+ - d_+^2 = 0$, $b_+c_+ - e_+^2 = 0$ and $d_+ = e_+$.

When $a_+ \gg c_+$, if $6J_1 - 10J_1\rho_{1a} - \frac{8}{3}J_1\rho_{1b} = 3(b_+ - 32d_+) < 0$ and no special cancelation between b_+ and $32d_+$, then $b_+ - 32d_+ = \mathcal{O}(-32d_+)$ and $b_+ \leq \mathcal{O}(32d_+) \ll 4a_+$. So, using Eq. (234),

$$\begin{aligned}
\frac{1}{256 \times 3^2} \frac{(6J_1 - 10J_1\rho_{1a} - \frac{8}{3}J_1\rho_{1b})^2}{3J_1 - 3J_1\rho_{1a} - 2J_1\rho_{1b}} \\
= \frac{(b_+ - 32d_+)^2}{256(4a_+ + b_+)} \leq \mathcal{O}\left(\frac{32^2 d_+^2}{256 \times 4a_+}\right) \leq \mathcal{O}(c_+).
\end{aligned} \tag{35}$$

We note here that $4a_+ + b_+$ is similar to $4(a_+ - 2e_+)$ and $4(a_+ + 8d_+)$ in this situation. So, we have similar result if the denominator of left hand side of Eq. (35), $3J_1 - 3J_1\rho_{1a} - 2J_1\rho_{1b} = 4a_+ + b_+$ is exchanged by $3J_1 - 3J_1\rho_{1a} - \frac{22}{9}J_1\rho_{1b} = 4(a_+ - 2e_+)$ or $(3J_1 + J_1\rho_{1a} - \frac{10}{3}J_1\rho_{1b})/3 = 4(a_+ + 8d_+)$.

Similarly, when $b_+ \gg c_+$, if $3J_1 - 3J_1\rho_{1a} - \frac{22}{9}J_1\rho_{1b} = 4(a_+ - 2e_+) < 0$ and no special cancelation between a_+ and $2e_+$, then $a_+ - 2e_+ = \mathcal{O}(-2e_+)$ and $a_+ \leq \mathcal{O}(2e_+) \ll b_+/4$. So, using Eq. (236),

$$\begin{aligned}
\frac{1}{8^2} \frac{(3J_1 - 3J_1\rho_{1a} - \frac{22}{9}J_1\rho_{1b})^2}{3J_1 - 3J_1\rho_{1a} - 2J_1\rho_{1b}} \\
= \frac{(a_+ - 2e_+)^2}{4(4a_+ + b_+)} \leq \mathcal{O}\left(\frac{2^2 e_+^2}{4b_+}\right) \leq \mathcal{O}(c_+).
\end{aligned} \tag{36}$$

We note here that $a_+ + b_+/4$ is similar to $a_+ - 2e_+$ and $a_+ + 8d_+$ in this situation. So, we have similar result if the denominator of left hand side in Eq. (36), $3J_1 - 3J_1\rho_{1a} - 2J_1\rho_{1b} = 4a_+ + b_+$ is exchanged by $\frac{4}{9}J_1\rho_{1b} = b_+ + 8e_+$ or $(6J_1 - 10J_1\rho_{1a} - \frac{8}{3}J_1\rho_{1b})/3 = b_+ - 32d_+$.

If we suppose that $b_+ = 0$ from some special models or other experimental results, we have c_+ lower limit from (23) and (234),

$$\frac{1}{1728} \frac{(3J_1 - 5J_1\rho_{1a})^2}{J_1 - J_1\rho_{1a}} = \frac{d_+^2}{a_+} \leq c_+. \tag{37}$$

Similarly, from (22) and (236), if we suppose that $a_+ = 0$ from some special models or other experimental results,

$$\frac{1}{1080} \frac{(3J_1 - 14J_1\rho_{1a})^2}{J_1 - 3J_1\rho_{1a}} = \frac{e_+^2}{b_+} \leq c_+. \quad (38)$$

4.5 A bound for d_+

4.5.1 Without using information c_+

We obtain a bound for d_+ using information from $G_0(x_1)$ measurement without using information on c_+ .

From the Eq. (14),

$$\frac{1}{36} \frac{9 + 3\rho_{1a} - 10\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} = \frac{8}{4a_+ + b_+} d_+ + \frac{a_+}{4a_+ + b_+}. \quad (39)$$

If $d_+ = 0$,

$$0 \leq \frac{1}{36} \frac{9 + 3\rho_{1a} - 10\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} \leq \frac{1}{4} \quad (40)$$

since $a_+, b_+ \geq 0$. So, if

$$\frac{9 + 3\rho_{1a} - 10\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} < 0, \quad (41)$$

then $d_+ < 0$; if

$$\frac{9 + 3\rho_{1a} - 10\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} > 9, \quad (42)$$

then $d_+ > 0$; and if

$$0 \leq \frac{9 + 3\rho_{1a} - 10\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} \leq 9, \quad (43)$$

then, from Eq. (33),

$$-J_1 \frac{3 - 3\rho_{1a} - 2\rho_{1b}}{32} \leq d_+ \leq J_1 \frac{3 - 3\rho_{1a} - 2\rho_{1b}}{32}. \quad (44)$$

Furthermore, we can determine d_+ allowed region as follows. Using the fact

$$0 \leq \frac{a_+}{4a_+ + b_+} \leq \frac{1}{4}, \quad (45)$$

which become the equation when $a_+ = 0$ and $b_+ = 0$, respectively,

$$\frac{8}{4a_+ + b_+} d_+ \leq \frac{1}{36} \frac{9 + 3\rho_{1a} - 10\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} \leq \frac{8}{4a_+ + b_+} d_+ + \frac{1}{4}. \quad (46)$$

The solution of this inequality about d_+ becomes

$$\frac{-9J_1 + 15J_1\rho_{1a} + 4J_1\rho_{1b}}{144} \leq d_+ \leq \frac{9J_1 + 3J_1\rho_{1a} - 10J_1\rho_{1b}}{288}. \quad (47)$$

This becomes equality, when $b_+ = 0$ and $a_+ = 0$, respectively.

4.5.2 Using information c_+

We obtain a bound for d_+ using information from $G_0(x_1)$ measurement with using information on c_+ .

From Eq. (14),

$$a_+c_+ + 8d_+c_+ = \frac{J_1}{36}(9 + 3\rho_{1a} - 10\rho_{1b})c_+. \quad (48)$$

This becomes

$$\frac{J_1}{36}(9 + 3\rho_{1a} - 10\rho_{1b})c_+ \geq d_+^2 + 8d_+c_+ \quad (49)$$

since $a_+c_+ - d_+^2 \geq 0$ as explained in appendix D. Using $c_+ > 0$,

$$\frac{J_1}{36c_+}(9 + 3\rho_{1a} - 10\rho_{1b}) \geq \left(\frac{d_+}{c_+} + 4\right)^2 - 16. \quad (50)$$

The solution of this inequality about d_+ becomes

$$\begin{aligned} -4c_+ - c_+ \sqrt{\frac{J_1}{36c_+}(9 + 3\rho_{1a} - 10\rho_{1b}) + 16} \\ \leq d_+ \leq -4c_+ + c_+ \sqrt{\frac{J_1}{36c_+}(9 + 3\rho_{1a} - 10\rho_{1b}) + 16}. \end{aligned} \quad (51)$$

So we have another limit of d_+ .

4.6 A bound for e_+

4.6.1 Without using information c_+

We obtain a bound for e_+ using information from $G_0(x_1)$ measurement without using information on c_+ .

Similar to the subsubsection 4.5.1,

$$\frac{4}{9} \frac{\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} = \frac{8}{4a_+ + b_+} e_+ + \frac{b_+}{4a_+ + b_+}, \quad (52)$$

and if $e_+ = 0$,

$$0 \leq \frac{4}{9} \frac{\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} \leq 1. \quad (53)$$

This becomes equation when $b_+ = 0$ and $a_+ = 0$ respectively. So, if

$$\frac{\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} < 0, \quad (54)$$

then $e_+ < 0$; if

$$\frac{\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} > \frac{9}{4}, \quad (55)$$

then $e_+ > 0$; and if

$$0 \leq \frac{\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} \leq \frac{9}{4}, \quad (56)$$

then, from Eq. (33),

$$-J_1 \frac{3 - 3\rho_{1a} - 2\rho_{1b}}{8} \leq e_+ \leq J_1 \frac{3 - 3\rho_{1a} - 2\rho_{1b}}{8}. \quad (57)$$

Furthermore, we can determine e_+ allowed region as follows. Using the fact

$$0 \leq \frac{b_+}{4a_+ + b_+} \leq 1, \quad (58)$$

which become the equation when $b_+ = 0$ and $a_+ = 0$, respectively,

$$\frac{8}{4a_+ + b_+} e_+ \leq \frac{4}{9} \frac{\rho_{1b}}{3 - 3\rho_{1a} - 2\rho_{1b}} \leq \frac{8}{4a_+ + b_+} e_+ + 1. \quad (59)$$

The solution of this inequality about e_+ becomes

$$\frac{J_1 \rho_{1b}}{18} - \frac{3J_1 - 3J_1 \rho_{1a} - 2J_1 \rho_{1b}}{8} \leq e \leq \frac{J_1 \rho_{1b}}{18}. \quad (60)$$

This becomes equality, when $a_+ = 0$ and $b_+ = 0$, respectively.

4.6.2 Using information c_+

We obtain a bound for e_+ using information from $G_0(x_1)$ measurement with using information on c_+ .

Similar to previous subsection, from Eq. (14),

$$b_+ c_+ + 8e_+ c_+ = \frac{4}{9} J_1 \rho_{1b} c_+. \quad (61)$$

This becomes

$$\frac{4}{9} J_1 \rho_{1b} c_+ \geq e_+^2 + 8e_+ c_+ \quad (62)$$

since $b_+ c_+ - e_+^2 \geq 0$ as explained in appendix D. Using $c_+ > 0$,

$$\frac{4}{9} \frac{J_1 \rho_{1b}}{c_+} \geq \left(\frac{e_+}{c_+} + 4 \right)^2 - 16. \quad (63)$$

The solution of this inequality about e_+ becomes

$$-4c_+ - 2c_+ \sqrt{\frac{J_1 \rho_{1b}}{9c_+} + 4} \leq e_+ \leq -4c_+ + 2c_+ \sqrt{\frac{J_1 \rho_{1b}}{9c_+} + 4}. \quad (64)$$

So we have another limit of e_+ .

5 $G_0(x_3)$: Energy Dependence of μ_3

In the previous section, we have discussed how the values of parameters c_+ , J_1 , ρ_{1a} and ρ_{1b} can be obtained from G_0 but we have not determined the values of a_+ , b_+ , d_+ and e_+ . This will be the subject of this section.

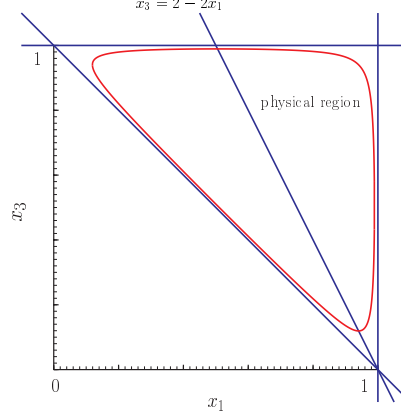


Figure 6: The physical region for $\tau^+ \rightarrow \mu_1 \mu_2 \mu_3$ is the area which is included in curbed line and besides the right side of the line $x_3 = 2 - 2x_1$. Near the line $x_1 = 1$, the differential branching ratio is singular since the intermediate state photon becomes real.

In the previous section, we gave x_1 dependence for G_0 . In this section, to determine the values of a_+ , b_+ , d_+ and e_+ , we give x_3 dependence of differential branching ratio. As in the previous section, we do not discuss the polarizations. For most of the analysis, muon mass dependence can be neglected. But, as we shall see, in some part of the phase space, m_μ dependence must be taken into account.

To study the formula of the G_0 as function of x_3 , we give the following prescription. We can get the x_3 dependence of $G_0(x_1, x_2)$ by using $x_1 + x_2 + x_3 = 2$. From the condition $x_1 \geq x_2$ and the relation $x_1 + x_2 + x_3 = 2$,

$$x_1 \geq 1 - \frac{x_3}{2}. \quad (65)$$

So, if all muon masses can be neglected, we have $1/2 \leq x_1 \leq 1$ and $0 \leq x_3 \leq 1$ as shown in Fig. 6. The integration of x_1 in the x_1 - x_3 plane is given by

$$\int_{1-\frac{x_3}{2}}^1 dx_1 G_0(x_1, x_3). \quad (66)$$

When $x_1 = 1$ in massless limit of muons, the photon in Fig. 1 becomes on shell and the propagator becomes singular. This leads a divergence of $x_1 = 1$ in differential branching ratio for $\tau \rightarrow 3\mu$ decay.

So we have to be more careful with the range of x_1 including the muon mass dependence. The on shell constraint $p_2^2 = m_\mu^2$ leads to

$$\frac{\delta^2}{4} + 1 - x_1 - x_3 + \frac{x_1 x_3}{2} - \frac{1}{2} \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3 \sqrt{x_1^2 - \delta^2} \sqrt{x_3^2 - \delta^2} = 0, \quad (67)$$

where $\delta \equiv 2m_\mu/m_\tau$. The true integration range for x_1 is given by Eqs. (65) and (67) with $-1 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3 \leq 1$ as shown in Fig. 6. We find it convenient to approximate the domain of integration as $1 - \frac{x_3}{2} \leq x_1 \leq 1 - (\frac{4\delta}{3})^2$. Here, the upper bound of x_1 is decided as follows. First, we calculate the total branching ratio of c_+ sector integrated on the true physical region in $\mathcal{O}(\delta^2)$. Next, we calculate the total branching ratio of c_+ sector integrated on the approximated region. Finally, we set the approximated upper bound to match these two total branching ratios. This approximation allows us to avoid the divergent region. With this approximation, we give the x_3 dependence of G_0 as

$$\begin{aligned} G_0(x_3) &= \int_{1-\frac{x_3}{2}}^{1-(\frac{4\delta}{3})^2} dx_1 G_0(x_1, x_3) \\ &= \frac{1}{2} J_3 x_3 \left\{ x_3(1-x_3) + \rho_{3a} x_3 \left(x_3 - \frac{2}{3}\right) + \rho_{3b} (x_3 - 1) \left(x_3 - \frac{1}{3}\right) \right\} \\ &\quad + 4c_+ \left\{ 2x_3 - 3x_3^2 + 2(2x_3^2 - 2x_3 + 1) \log \left[\frac{3\sqrt{x_3}}{4\delta} \right] \right\}, \end{aligned} \quad (68)$$

where

$$\begin{aligned} J_3 &= \frac{1}{3} (12a_+ + 5b_+ + 144d_+ + 36e_+) \\ J_3 \rho_{3a} &= b_+ + 12e_+ \\ J_3 \rho_{3b} &= 48d_+. \end{aligned} \quad (69)$$

First, to determine c_+ , it is convenient to define the function

$$F_3(x_3) = \frac{G_0(x_3)}{8 \log \sqrt{x_3}}. \quad (70)$$

By choosing the kinematics such that $x_3 \rightarrow 0$ for the function $F_3(x_3)$, we can obtain c_+ as

$$F_3(x_3)|_{x_3 \rightarrow 0} = c_+. \quad (71)$$

Next, we define the function

$$\begin{aligned} F_4(x_3) &= \frac{G_0(x_3) - (c_+, x_3 \text{ term})}{x_3} \\ &= \frac{1}{2} J_3 \left\{ x_3(1-x_3) + \rho_{3a} x_3 \left(x_3 - \frac{2}{3}\right) + \rho_{3b} (x_3 - 1) \left(x_3 - \frac{1}{3}\right) \right\}, \end{aligned} \quad (72)$$

where

$$(c_+, x_3 \text{ term}) = 4c_+ \left\{ 2x_3 - 3x_3^2 + 2(2x_3^2 - 2x_3 + 1) \log \left[\frac{3\sqrt{x_3}}{4\delta^2} \right] \right\}. \quad (73)$$

From the function $F_4(x_3)$, we can obtain the values of J_3 , ρ_{3a} and ρ_{3b} by fitting the experimental data as follows:

$$\begin{aligned} \int_0^1 dx_3 F_4(x_3) &= \frac{1}{12} J_3 \\ F_4(x_3)|_{x_3=1} &= \frac{1}{6} J_3 \rho_{3a} \\ F_4(x_3)|_{x_3=0} &= \frac{1}{6} J_3 \rho_{3b}. \end{aligned} \quad (74)$$

In addition to J_1 , ρ_{1a} and ρ_{1b} from previous section, the parameters a_+ , b_+ , d_+ and e_+ can now be determined. For example,

$$\begin{aligned} a_+ &= \frac{1}{4} J_3 \left(1 + \frac{1}{3} \rho_{3a} - \rho_{3b} \right) - \frac{2}{9} J_1 \rho_{1b} \\ b_+ &= \frac{4}{3} J_1 \rho_{1b} - 2 J_3 \rho_{3a} \\ d_+ &= \frac{1}{48} J_3 \rho_{3b} \\ e_+ &= \frac{1}{4} J_3 \rho_{3a} - \frac{1}{9} J_1 \rho_{1b}. \end{aligned} \quad (75)$$

From the differential cross section which are the functions of x_1 and x_2 , x_1 and x_3 or x_2 and x_3 , all we can determine are the quantities a_+ , b_+ , c_+ , d_+ , e_+ . Now, we will discuss some specific cases in which we can go further.

6 Physics from $G_0(x_3)$ Distribution of μ_3

Similar to the section 4, we have some bounds for a_+ , b_+ , d_+ and e_+ from the analysis of previous section.

6.1 What can we conclude if one of a_+ , b_+ , c_+ , d_+ , e_+ = 0

If we impose that one of a_+ , b_+ , c_+ , d_+ and e_+ is zero from other experimental results or supposing some specific models, the results are as follows. When $a_+ = 0$, also $d_+ = 0$ and

$$\begin{aligned} J_3 &= \frac{1}{3} (5b_+ + 36e_+) \\ J_3 \rho_{3a} &= b_+ + 12e_+ \\ J_3 \rho_{3b} &= 0. \end{aligned} \quad (76)$$

When $b_+ = 0$, also $e_+ = 0$ and

$$\begin{aligned} J_3 &= 4(a_+ + 12d_+) \\ J_3\rho_{3a} &= 0 \\ J_3\rho_{3b} &= 48d_+. \end{aligned} \tag{77}$$

When $c_+ = 0$, also $d_+ = e_+ = 0$ and

$$\begin{aligned} J_3 &= \frac{1}{3}(12a_+ + 5b_+) \\ J_3\rho_{3a} &= b_+ \\ J_3\rho_{3b} &= 0. \end{aligned} \tag{78}$$

When $d_+ = 0$,

$$\begin{aligned} J_3 &= \frac{1}{3}(12a_+ + 5b_+ + 36e_+) \\ J_3\rho_{3a} &= b_+ + 12e_+ \\ J_3\rho_{3b} &= 0. \end{aligned} \tag{79}$$

When $e_+ = 0$,

$$\begin{aligned} J_3 &= \frac{1}{3}(12a_+ + 5b_+ + 144d_+) \\ J_3\rho_{3a} &= b_+ \\ J_3\rho_{3b} &= 48d_+. \end{aligned} \tag{80}$$

If any one of a_+ , b_+ , c_+ or e_+ is zero, we can determine a_+ , b_+ , c_+ , d_+ and e_+ from only the x_3 distribution. However, if $d_+ = 0$, then we cannot determine a_+ , b_+ and e_+ , independently.

6.2 Results if either $a_+ \neq 0$, $b_+ \neq 0$, $d_+ \neq 0$ or $e_+ \neq 0$

Similarly, When only $a_+ \neq 0$,

$$\begin{aligned} J_3 &= 4a_+ \\ \rho_{3a} &= 0 \\ \rho_{3b} &= 0. \end{aligned} \tag{81}$$

When only $b_+ \neq 0$,

$$\begin{aligned} J_3 &= \frac{5}{3}b_+ \\ \rho_{3a} &= \frac{3}{5} \\ \rho_{3b} &= 0. \end{aligned} \tag{82}$$

When only a_+ , c_+ , $d_+ \neq 0$ but a_+ is negligible,

$$\begin{aligned} J_3 &= 48d_+ \\ \rho_{3a} &= 0 \\ \rho_{3b} &= 1. \end{aligned} \tag{83}$$

When only b_+ , c_+ , $e_+ \neq 0$ but b_+ is negligible,

$$\begin{aligned} J_3 &= 12e_+ \\ J_3\rho_{3a} &= 1 \\ J_3\rho_{3b} &= 0. \end{aligned} \tag{84}$$

In these case, ρ_{3a} and ρ_{3b} are independent from a_+ , b_+ , d_+ and e_+ .

The results in subsection 4.2 and here is summed in the table below.

Table 1: values of J' s and ρ 's in some cases

	J_1	ρ_{1a}	ρ_{1b}	J_3	ρ_{3a}	ρ_{3b}
(A)	$\frac{10}{3}a_+$	$\frac{3}{5}$	0	$4a_+$	0	0
(B)	$\frac{7}{3}b_+$	$\frac{3}{14}$	$\frac{27}{28}$	$\frac{5}{3}b_+$	$\frac{3}{5}$	0
(C)	0	—	—	0	—	—
(D)	$24d_+$	1	0	$48d_+$	0	1
(E)	$18e_+$	$\frac{1}{3}$	1	$12e_+$	1	0

(A): all but g_1, g_2, g_3, g_4 are vanishing

(B): all but g_5, g_6 are vanishing

(C): all but eA_R, eA_L are vanishing

(D): all but $Re[g_3eA_L^*] + Re[g_4eA_R^*], eA_R, eA_L$ are vanishing

(E): all but $Re[g_5eA_L^*] + Re[g_6eA_R^*], eA_R, eA_L$ are vanishing

6.3 What happens if $\tau \rightarrow \mu\gamma$ dominates

Similar to subsection 4.3, in this case,

$$\begin{aligned} J_3 &= 12(4d_+ + e_+) \\ J_3\rho_{3a} &= 12e_+ \\ J_3\rho_{3b} &= 48d_+. \end{aligned} \tag{85}$$

From Eqs. (234) and (236) in appendix D,

$$\begin{aligned} a_+ &\geq \frac{d_+^2}{c_+} = \frac{1}{c_+} \left(\frac{J_3 \rho_{3b}}{48} \right)^2 \\ b_+ &\geq \frac{e_+^2}{c_+} = \frac{1}{c_+} \left(\frac{J_3 \rho_{3a}}{12} \right)^2. \end{aligned} \quad (86)$$

So, we can determine a_+ and b_+ lower limit, though we can't determine a_+ and b_+ , directly.

We note here that since the relation $J_3 \rho_{3b} = 48d_+$ is always true, a_+ lower limit is always determined.

6.4 Lower limit of c_+ when $\tau \rightarrow \mu\gamma$ is highly suppressed

We give three types of c_+ lower limits. Especially, these are very useful when $\tau \rightarrow \mu\gamma$ is highly suppressed compared with $\tau \rightarrow 3\mu$ and c_+ cannot determine directly.

First, we give the relations,

$$\begin{aligned} 3J_3 - 5J_3 \rho_{3a} - 3J_3 \rho_{3b} &= 12(a_+ - 2e_+) \\ 3J_3 - 3J_3 \rho_{3a} - 3J_3 \rho_{3b} &= 2(6a_+ + b_+) \\ J_3 \rho_{3a} &= b_+ + 12e_+ \\ J_3 \rho_{3b} &= 48d_+ \end{aligned} \quad (87)$$

from Eqs. (69).

Using Eqs. (87) and (234) and the fact that $a_+, b_+ \geq 0$,

$$\frac{(J_3 \rho_{3b})^2}{576(J_3 - J_3 \rho_{3a} - J_3 \rho_{3b})} = \frac{6d_+^2}{6a_+ + b_+} \leq \frac{d_+^2}{a_+} \leq c_+. \quad (88)$$

Using Eqs. (87),

$$\frac{(J_3 \rho_{3b})^2}{192(3J_3 - 5J_3 \rho_{3a} - 3J_3 \rho_{3b})} = \frac{d_+^2}{a_+ - 2e_+}. \quad (89)$$

Here, if $c_+ \ll b_+$, then $J_3 \rho_{3a} = b_+ + 12e_+ = \mathcal{O}(b_+)$ and $b_+ \geq \mathcal{O}(|12e_+|)$. So, if $(3J_3 - 5J_3 \rho_{3a} - 3J_3 \rho_{3b})/12 \gtrsim J_3 \rho_{3a}$, this means $a_+ - 2e_+ \geq \mathcal{O}(b_+)$ and then $a_+ - 2e_+ \simeq a_+$. So, using the relation (234),

$$\frac{(J_3 \rho_{3b})^2}{192(3J_3 - 5J_3 \rho_{3a} - 3J_3 \rho_{3b})} \lesssim c_+. \quad (90)$$

If

$$3J_3 - 5J_3 \rho_{3a} - 3J_3 \rho_{3b} = 12(a_+ - 2e_+) < 0, \quad (91)$$

$b_+ \gg c_+$ and there is no special cancelation between a_+ and $2e_+$, then $a_+ - 2e_+ = \mathcal{O}(-2e_+)$ and $a_+ \leq \mathcal{O}(2e_+) \ll b_+/6$. So,

$$\frac{(3J_3 - 5J_3\rho_{3a} - 3J_3\rho_{3b})^2}{864(J_3 - J_3\rho_{3a} - J_3\rho_{3b})} = \frac{(a_+ - 2e_+)^2}{4(6a_+ + b_+)} = \mathcal{O}\left(\frac{e_+^2}{b_+}\right) \leq \mathcal{O}(c_+). \quad (92)$$

We note here that $6a_+ + b_+$ is similar to $b_+ + 12e_+$ in this situation. So, we have similar result if the denominator of left hand side in Eq. (92), $J_3 - J_3\rho_{3a} - J_3\rho_{3b} = 2(6a_+ + b_+)/3$ is exchanged by $2J_3\rho_{3a}/3 = 2(b_+ + 12e_+)/3$.

From (77) and (234), if we suppose that $b_+ = 0$ from some special models or other experimental results,

$$\frac{1}{24^2} \frac{(J_3\rho_{3b})^2}{J_3 - J_3\rho_b} = \frac{d_+^2}{a_+} \leq c_+. \quad (93)$$

Similarly, from (76) and (236), if we suppose that $a_+ = 0$ from some special models or other experimental results,

$$\frac{1}{864} \frac{(3J_3 - 5J_3\rho_{1a})^2}{J_3 - J_3\rho_{1a}} = \frac{e_+^2}{b_+} \leq c_+. \quad (94)$$

6.5 A bound for e_+

6.5.1 Without using information c_+

We obtain a bound for e_+ using information from $G_0(x_1)$ measurement without using information on c_+ .

Similarly to the subsection 4.6,

$$\frac{2\rho_{3a}}{3 - 3\rho_{3a} - 3\rho_{3b}} = \frac{12}{6a_+ + b_+} e_+ + \frac{b_+}{6a_+ + b_+}, \quad (95)$$

and if $e_+ = 0$,

$$0 \leq \frac{2\rho_{3a}}{3 - 3\rho_{3a} - 3\rho_{3b}} \leq 1. \quad (96)$$

This becomes equation when $b_+ = 0$ and $a_+ = 0$ respectively. So, if

$$\frac{\rho_{3a}}{1 - \rho_{3a} - \rho_{3b}} < 0, \quad (97)$$

then $e_+ < 0$; if

$$\frac{\rho_{3a}}{1 - \rho_{3a} - \rho_{3b}} > \frac{3}{2}, \quad (98)$$

then $e_+ > 0$; and if

$$0 \leq \frac{\rho_{3a}}{1 - \rho_{3a} - \rho_{3b}} \leq \frac{3}{2}, \quad (99)$$

then, from Eqs. (87),

$$-J_3 \frac{1 - \rho_{3a} - \rho_{3b}}{8} \leq e_+ \leq J_3 \frac{1 - \rho_{3a} - \rho_{3b}}{8}. \quad (100)$$

Furthermore, we can determine e_+ allowed region as follows. Using the fact

$$0 \leq \frac{b_+}{6a_+ + b_+} \leq 1, \quad (101)$$

which become the equation when $b_+ = 0$ and $a_+ = 0$, respectively,

$$\frac{12}{6a_+ + b_+} e_+ \leq \frac{2\rho_{3a}}{3 - 3\rho_{3a} - 3\rho_{3b}} \leq \frac{12}{6a_+ + b_+} e_+ + 1. \quad (102)$$

The solution of this inequality about e_+ becomes

$$\frac{-3J_3 + 5J_3\rho_{3a} + 3J_3\rho_{3b}}{24} \leq e_+ \leq \frac{J_3\rho_{3a}}{12}. \quad (103)$$

This becomes equality, when $a_+ = 0$ and $b_+ = 0$, respectively.

6.5.2 Using information c_+

We obtain a bound for e_+ using information from $G_0(x_1)$ measurement with using information on c_+ .

Similar to subsection 4.6.2, from Eq. (69),

$$b_+c_+ + 12e_+c_+ = J_3\rho_{3b}c_+. \quad (104)$$

This becomes

$$J_3\rho_{3b}c_+ \geq e_+^2 + 12e_+c_+ \quad (105)$$

since $b_+c_+ - e_+^2 \geq 0$ as explained in appendix D. Using $c_+ > 0$,

$$\frac{J_3\rho_{3b}}{c_+} \geq \left(\frac{e_+}{c_+} + 6 \right)^2 - 36. \quad (106)$$

The solution of this inequality about e_+ becomes

$$-6c_+ - c_+ \sqrt{\frac{J_3\rho_{3b}}{c_+} + 36} \leq e_+ \leq -6c_+ + c_+ \sqrt{\frac{J_3\rho_{3b}}{c_+} + 36} \quad (107)$$

This becomes equality when $b_+c_+ - e_+^2 = 0$. So we have another limit of e_+ .

7 Physics Implications from $G_0(x_1)$ and $G_0(x_3)$ Distributions

From the analysis of both of $G_0(x_1)$ and $G_0(x_3)$ distributions, we can determine not only a_+ , b_+ , c_+ , d_+ and e_+ but also in more detail in some suitable cases as explained in this section.

Introducing new real parameters $\{r_1, r_2, r_3, r_4, r_5, r_6, r_R, r_L\} \geq 0$ and $2\pi > \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_R, \theta_L\} \geq 0$, the effective coupling constants can be explained as

$$\begin{aligned} g_1 &= r_1 e^{i\theta_1} & g_2 &= r_2 e^{i\theta_2} \\ g_3 &= r_3 e^{i\theta_3} & g_4 &= r_4 e^{i\theta_4} \\ g_5 &= r_5 e^{i\theta_5} & g_6 &= r_6 e^{i\theta_6} \\ eA_R &= r_R e^{i\theta_R} & eA_L &= r_L e^{i\theta_L}. \end{aligned} \quad (108)$$

Then,

$$\begin{aligned} a_+ &= \frac{1}{16}(r_1^2 + r_2^2) + r_3^2 + r_4^2 \\ b_+ &= r_5^2 + r_6^2 \\ c_+ &= r_R^2 + r_L^2 \\ d_+ &= -r_3 r_L \cos(\theta_3 - \theta_L) - r_4 r_R \cos(\theta_4 - \theta_R) \\ e_+ &= -r_6 r_R \cos(\theta_6 - \theta_R) - r_5 r_L \cos(\theta_5 - \theta_L). \end{aligned} \quad (109)$$

7.1 What can we say if $a_+ c_+ - d_+^2 = 0$

From Eq. (234) in appendix D, $a_+ c_+ - d_+^2 = 0$ only if

$$\begin{cases} g_3 eA_R = g_4 eA_L & (110) \\ \text{Im}[g_3 eA_L^*] + \text{Im}[g_4 eA_R^*] = 0 & (111) \\ g_1 = g_2 = 0. & (112) \end{cases}$$

In that case,

$$r_3 r_R e^{i(\theta_3 + \theta_R)} = r_4 r_L e^{i(\theta_4 + \theta_L)} \quad (113)$$

from the relation (110). This means

$$\begin{cases} r_3 r_R = r_4 r_L \\ \theta_3 - \theta_L = \theta_4 - \theta_R, \theta_4 - \theta_R \pm 2\pi. \end{cases} \quad (114)$$

Then, from the relation (111),

$$\begin{aligned} & r_3 r_L \sin(\theta_3 - \theta_L) + r_4 r_R \sin(\theta_4 - \theta_R) \\ &= r_3 r_L \sin(\theta_3 - \theta_L) + r_4 r_R \sin(\theta_3 - \theta_L) \\ &= (r_3 r_L + r_4 r_R) \sin(\theta_3 - \theta_L) = 0. \end{aligned} \quad (115)$$

This means

$$\begin{aligned} r_3 r_L + r_4 r_R &= 0 \\ \text{and/or} \\ \theta_3 - \theta_L &= 0, \pm\pi. \end{aligned} \quad (116)$$

Here, $r_3 r_L + r_4 r_R \neq 0$ because of the conditions (235) and (112). Then $\theta_3 - \theta_L = \theta_4 - \theta_R = 0, \pm\pi$ and

$$d_+ = \mp(r_3 r_L + r_4 r_R). \quad (117)$$

The sign of right hand side is minus if $\theta_3 - \theta_L = \theta_4 - \theta_R = 0$ and plus if $\theta_3 - \theta_L = \theta_4 - \theta_R = \pm\pi$.

In these cases,

$$\begin{cases} a_+ = r_3^2 + r_4^2 \\ c_+ = r_R^2 + r_L^2 \\ d_+ = \mp(r_3 r_L + r_4 r_R) \\ r_3 r_R = r_4 r_L. \end{cases} \quad (118)$$

One of these conditions is dependent on others. For example, one condition $c_+ = r_R^2 + r_L^2$ can be expressed using other conditions as

$$\begin{cases} \frac{d_+^2}{a_+} = \frac{(r_3 r_L + r_4 r_R)^2}{r_3^2 + r_4^2} = \frac{(\frac{r_4 r_L}{r_R} r_L + r_4 r_R)^2}{\frac{r_4^2 r_L^2}{r_R^2} + r_4^2} = r_R^2 + r_L^2 \\ \frac{d_+^2}{a_+} = c_+ \end{cases} \quad (119)$$

since $a_+ c_+ - d_+^2 = 0$. So, there are only three independent conditions in (118). Using one of r_3, r_4, r_R and r_L , we can express others. For instance, if we know r_R from other experiments or some special models, we can represent other coupling constants as

$$\begin{aligned} r_L &= \sqrt{c_+ - r_R^2} \\ r_4 &= \frac{|d_+| r_R}{c_+} \\ r_3 &= \sqrt{a_+} \sqrt{1 - \frac{r_R^2}{c_+}}. \end{aligned} \quad (120)$$

Here, we note that if $\{r_3, r_4, r_R, r_L\} \neq 0$,

$$\begin{aligned} \frac{a_+}{c_+} &= \frac{|g_3|^2}{|eA_L|^2} = \frac{|g_4|^2}{|eA_R|^2} \\ \frac{|d_+|}{c_+} &= \frac{|g_3|}{|eA_L|} = \frac{|g_4|}{|eA_R|} \end{aligned} \quad (121)$$

from the relation (110).

If $r_3 = 0$, then $r_L = 0$ from the relations (235), (110) and (112). Similarly, $r_R = 0$ if $r_4 = 0$; $r_3 = 0$ if $r_L = 0$; and $r_4 = 0$ if $r_R = 0$.

7.2 What if it so happens that $a_+c_+ - d_+^2 > 0$

When $a_+c_+ - d_+^2 > 0$, at least, one of the relations

$$\begin{aligned} g_3A_R - g_4A_L &\neq 0 \\ Im[g_3A_L^*] + Im[g_4A_R^*] &\neq 0 \\ g_1 &\neq 0 \\ g_2 &\neq 0 \end{aligned} \tag{122}$$

is satisfied.

First, if $g_3A_R - g_4A_L \neq 0$, then $g_3 \neq g_4$ and/or $A_R \neq A_L$. This leads parity violation which is defined in Appendix E. Next, if $Im[g_3A_L^*] + Im[g_4A_R^*] \neq 0$, then the relative phases of one of $g_3A_L^*$ and/or $g_4A_R^*$ have nonzero values. This leads CP violation. If $g_1 \neq 0$ or $g_2 \neq 0$, the scalar current and/or pseudo scalar current exists.

7.3 What can we say if $b_+c_+ - e_+^2 = 0$

From Eq. (236) in appendix D, $b_+c_+ - e_+^2 = 0$ only if

$$\begin{cases} g_5eA_R = g_6eA_L \\ Im[g_5eA_L^*] + Im[g_6eA_R^*] = 0 \end{cases} \tag{123}$$

$$\tag{124}$$

In that case,

$$r_5r_Re^{i(\theta_5+\theta_R)} = r_6r_L e^{i(\theta_6+\theta_L)} \tag{125}$$

from the relation (123). This means

$$\begin{cases} r_5r_R = r_6r_L \\ \theta_5 - \theta_L = \theta_6 - \theta_R, \theta_5 - \theta_R \pm 2\pi. \end{cases} \tag{126}$$

Then, from the relation (124),

$$\begin{aligned} &r_5r_L \sin(\theta_5 - \theta_L) + r_6r_R \sin(\theta_6 - \theta_R) \\ &= r_5r_L \sin(\theta_5 - \theta_L) + r_6r_R \sin(\theta_5 - \theta_L) \\ &= (r_5r_L + r_6r_R) \sin(\theta_5 - \theta_L) = 0. \end{aligned} \tag{127}$$

This means

$$\begin{aligned} r_5r_L + r_6r_R &= 0 \\ \text{and/or} \\ \theta_5 - \theta_L &= 0, \pm\pi. \end{aligned} \tag{128}$$

Here, $r_5 r_L + r_6 r_R \neq 0$ because of the conditions (237). Then $\theta_5 - \theta_L = \theta_6 - \theta_R = 0, \pm\pi$ and

$$e_+ = \mp(r_5 r_L + r_6 r_R). \quad (129)$$

The sign of right hand side is minus if $\theta_5 - \theta_L = \theta_6 - \theta_R = 0$ and plus if $\theta_5 - \theta_L = \theta_6 - \theta_R = \pm\pi$.

In these cases,

$$\begin{cases} b_+ = r_5^2 + r_6^2 \\ c_+ = r_R^2 + r_L^2 \\ e_+ = \mp(r_5 r_L + r_6 r_R) \\ r_5 r_R = r_6 r_L. \end{cases} \quad (130)$$

One of these conditions is dependent on others. For example, one condition $c_+ = r_R^2 + r_L^2$ can be expressed using other conditions as

$$\begin{cases} \frac{e_+^2}{b_+} = \frac{(r_5 r_L + r_6 r_R)^2}{r_5^2 + r_6^2} = \frac{(\frac{r_6 r_L}{r_R} r_L + r_6 r_R)^2}{\frac{r_6^2 r_L^2}{r_R^2} + r_6^2} = r_R^2 + r_L^2 \\ \frac{e_+^2}{b_+} = c_+ \end{cases} \quad (131)$$

since $b_+ c_+ - e_+^2 = 0$. So, there are only three independent conditions in (130). Using one of r_5, r_6, r_R and r_L , we can express others. For instance, if we know r_R from other experiments or some special models, we can represent other coupling constants as

$$\begin{aligned} r_L &= \sqrt{c_+ - r_R^2} \\ r_6 &= \frac{|e_+| r_R}{c_+} \\ r_5 &= \sqrt{b_+} \sqrt{1 - \frac{r_R^2}{c_+}}. \end{aligned} \quad (132)$$

Here, we note that if $\{r_5, r_6, r_R, r_L\} \neq 0$,

$$\begin{aligned} \frac{b_+}{c_+} &= \frac{|g_5|^2}{|e A_L|^2} = \frac{|g_6|^2}{|e A_R|^2} \\ \frac{|e_+|}{c_+} &= \frac{|g_5|}{|e A_L|} = \frac{|g_6|}{|e A_R|} \end{aligned} \quad (133)$$

from the relation (123).

If $r_5 = 0$, then $r_L = 0$ from the relations (237) and (123). Similarly, $r_R = 0$ if $r_6 = 0$; $r_5 = 0$ if $r_L = 0$; and $r_6 = 0$ if $r_R = 0$.

7.4 What if it so happens that $b_+c_+ - e_+^2 > 0$

When $b_+c_+ - e_+^2 > 0$, at least, one of the relations

$$\begin{aligned} g_5 A_R - g_6 A_L &\neq 0 \\ \text{Im}[g_5 A_L^*] + \text{Im}[g_6 A_R^*] &\neq 0 \end{aligned} \quad (134)$$

is satisfied.

First, if $g_5 A_R - g_6 A_L \neq 0$, then $g_5 \neq g_6$ and/or $A_R \neq A_L$. This leads parity violation which is defined in Appendix E. Next, if $\text{Im}[g_5 A_L^*] + \text{Im}[g_6 A_R^*] \neq 0$, then the relative phases of one of $g_5 A_L^*$ and/or $g_6 A_R^*$ have nonzero values. This leads CP violation.

7.5 What if one of $A_L, A_R, g_3, g_4, g_5, g_6 = 0$

We reveal here that even if $a_+c_+ - d_+^2 > 0$ or $b_+c_+ - e_+^2 > 0$, we can determine the lower limit of $|g_3|, |g_4|, |g_5|, |g_6|, |eA_R|$ and $|eA_L|$ in the case $A_L = 0, A_R = 0, A_L = 0, A_R = 0, g_3$ or $g_5 = 0$ and g_4 or $g_6 = 0$, respectively. If we restrict that $A_L = 0$ from other observation for example $\tau \rightarrow \mu\gamma$ decay or some specific models,

$$\begin{aligned} \frac{d_+^2}{c_+} &= r_4^2 \cos^2(\theta_4 - \theta_R) \leq |g_4|^2, \\ \frac{e_+^2}{c_+} &= r_6^2 \cos^2(\theta_6 - \theta_R) \leq |g_6|^2. \end{aligned} \quad (135)$$

So, the lower limit of $|g_4|, |g_6|$ is determined. Similarly, if $A_R = 0$,

$$\begin{aligned} \frac{d_+^2}{c_+} &= r_3^2 \cos^2(\theta_3 - \theta_L) \leq |g_3|^2, \\ \frac{e_+^2}{c_+} &= r_5^2 \cos^2(\theta_5 - \theta_L) \leq |g_5|^2; \end{aligned} \quad (136)$$

if $g_3 = 0$,

$$\frac{d_+^2}{a_+} = \frac{r_4^2}{r_4^2 + \frac{r_1^2 + r_2^2}{16}} r_R^2 \cos^2(\theta_4 - \theta_R) \leq r_R^2 \cos^2(\theta_4 - \theta_R) \leq |eA_R|^2; \quad (137)$$

if $g_5 = 0$,

$$\frac{e_+^2}{b_+} = r_R^2 \cos^2(\theta_6 - \theta_R) \leq |eA_R|^2; \quad (138)$$

and if $g_4 = 0$,

$$\frac{d_+^2}{a_+} = \frac{r_3^2}{r_3^2 + \frac{r_1^2 + r_2^2}{16}} r_L^2 \cos^2(\theta_3 - \theta_L) \leq r_L^2 \cos^2(\theta_3 - \theta_L) \leq |eA_L|^2; \quad (139)$$

if $g_6 = 0$,

$$\frac{e_+^2}{b_+} = r_L^2 \cos^2(\theta_5 - \theta_L) \leq |eA_L|^2. \quad (140)$$

The lower limits of $|g_4|$, $|g_6|$, $|g_3|$, $|g_5|$, $|eA_R|$ and $|eA_L|$ are determined in each case.

7.6 What can we say about parity and CP symmetries

If parity or charge symmetry exists,

$$\begin{aligned} r_1 &= r_2 \\ r_3 &= r_4 \\ r_5 &= r_6 \\ \theta_3 &= \pm\theta_4 \\ \theta_5 &= \pm\theta_6 \\ r_R &= r_L \\ \theta_R &= \pm\theta_L. \end{aligned} \quad (141)$$

So,

$$a_+c_+ - d_+^2 = \frac{r_1^2 r_L^2}{4} + 4r_3^2 r_L^2 \sin^2(\theta_3 - \theta_L) \quad (142)$$

from Eq. (234), and

$$b_+c_+ - e_+^2 = 4r_5^2 r_L^2 \sin^2(\theta_5 - \theta_L) \quad (143)$$

from Eq. (236). If $a_+c_+ - d_+^2 \neq 0$, then $|g_1| = |g_2| \neq 0$ and/or CP is violated. If $b_+c_+ - e_+^2 \neq 0$, then CP is violated.

If both of parity and CP symmetries exist,

$$\begin{aligned} r_1 &= r_2 \\ r_3 &= r_4 \\ r_5 &= r_6 \\ \theta_3 &= \theta_4 = 0, \pi \\ \theta_5 &= \theta_6 = 0, \pi \\ r_R &= r_L \\ \theta_R &= \theta_L = 0, \pi. \end{aligned} \quad (144)$$

So,

$$a_+c_+ - d_+^2 = \frac{r_1^2 r_L^2}{4} \quad (145)$$

from Eq. (234), and

$$b_+c_+ - e_+^2 = 0 \quad (146)$$

from Eq. (236). When $a_+c_+ - d_+^2 \neq 0$, $|g_1| = |g_2| \neq 0$.

7.7 What happens if $Br(\tau \rightarrow \mu\gamma)$ times the fine structure constant is much smaller than $Br(\tau \rightarrow 3\mu)$

In the case $c_+ \ll$ others, we can't determine c_+ directly. However, using the relations (234) and (236),

$$\begin{aligned} c_+ &\geq \frac{d_+^2}{a_+}, \\ c_+ &\geq \frac{e_+^2}{b_+}. \end{aligned} \tag{147}$$

So, we can determine c_+ lower limit.

This method is very useful to predict $\tau \rightarrow \mu\gamma$ branching ratio in the case that $\tau \rightarrow 3\mu$ is detected while $\tau \rightarrow \mu\gamma$ has not been detected, yet.

8 Angular Distribution, $G_i^s(x_1)$

As we shall see, angular distribution of the particle a in $\tau \rightarrow \nu_\tau + a + \text{anything}$ will give us information about a_- , b_- , c_- , d_- , e_- , f_+ and g_+ .

The general formula (9) contains all the information. However, it is too complex to analyze it here. So, we integrate about $d\Omega$ to simplify the formula. Furthermore, we introduce three formulae, which are the general formulae integrated about three kinds of azimuthal angles of the momentum \mathbf{k}_a as explained in Figs. 7-9 to simplify the formula, respectively. Thanks to this prescription, we obtain three simple formulae which are convenient to analyze here.

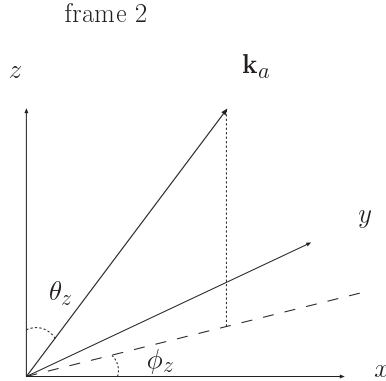


Figure 7: The polar coordinate of frame 2. The North pole is z direction.

Consider the double differential distribution, where d^3k_a is replaced by

$dk_a d \cos \theta_z d\phi_z$:

$$\begin{aligned}
& \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a d\phi_{az} d \cos \theta_{az}} \\
&= Br(\tau \rightarrow \mu \nu \bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{\alpha^2 \beta y_a^2}{\pi^2 q^2 \lambda_a} \\
&\quad \times \left[G_0(x_1, x_2) G_1^a(y_a) \left(2 + \frac{1}{\gamma^2}\right) \right. \\
&\quad \left. - \sum_i G_i^s(x_1, x_2) G_2^a(y_a) \left\{ \left(2 - \frac{1}{\gamma^2}\right) \hat{k}_{az} P_{iz} - \beta^2 \hat{k}_{ay} P_{iy} + \left(1 + \frac{1}{\gamma^2}\right) \hat{k}_{ax} P_{ix} \right\} \right], \tag{148}
\end{aligned}$$

where $y_a = 2E_a/m_\tau$. $\sum_i G_i^s(x_1, x_2) P_{iz}$, $\sum_i G_i^s(x_1, x_2) P_{ix}$, $\sum_i G_i^s(x_1, x_2) P_{iy}$ can be obtained by considering three single differential distributions as follows.

Integrate $d\phi_{az}$, then

$$\begin{aligned}
& \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a d \cos \theta_{az}} \\
&= 2\pi Br(\tau \rightarrow \mu \nu \bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{\alpha^2 \beta y_a^2}{\pi^2 q^2 \lambda_a} \\
&\quad \times \left[G_0(x_1, x_2) G_1^a(y_a) \left(2 + \frac{1}{\gamma^2}\right) \right. \\
&\quad \left. - \sum_i G_i^s(x_1, x_2) G_2^a(y_a) \left(2 - \frac{1}{\gamma^2}\right) \hat{k}_{az} P_{iz} \right]. \tag{149}
\end{aligned}$$

And using the equation

$$\begin{aligned}
& \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a} \\
&= 4\pi Br(\tau \rightarrow \mu \nu \bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{\alpha^2 \beta y_a^2}{\pi^2 q^2 \lambda_a} \\
&\quad \times G_0(x_1, x_2) G_1^a(y_a) \left(2 + \frac{1}{\gamma^2}\right), \tag{150}
\end{aligned}$$

then

$$\begin{aligned}
& \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a d \cos \theta_{az}} - \frac{1}{2} \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a} \\
&= -2\pi Br(\tau \rightarrow \mu \nu \bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{\alpha^2 \beta y_a^2}{\pi^2 q^2 \lambda_a} \\
&\quad \times G_2^a(y_a) \left(2 - \frac{1}{\gamma^2}\right) \hat{k}_{az} \sum_i G_i^s(x_1, x_2) P_{iz}. \tag{151}
\end{aligned}$$

Similarly, Taking the coordinates as Figs. 8 and 9, we have the equations

$$\begin{aligned} & \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a d \cos \theta_{ax}} - \frac{1}{2} \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a} \\ &= -2\pi Br(\tau \rightarrow \mu\nu\bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{\alpha^2 \beta y_a^2}{\pi^2 q^2 \lambda_a} \\ & \quad \times G_2^a(y_a) \left(1 + \frac{1}{\gamma^2}\right) \hat{k}_{ax} \sum_i G_i^s(x_1, x_2) P_{ix} \end{aligned} \quad (152)$$

and

$$\begin{aligned} & \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a d \cos \theta_{ay}} - \frac{1}{2} \frac{d\sigma}{dx_1 dx_2 d\Omega_\tau d\psi dy_a} \\ &= 2\pi Br(\tau \rightarrow \mu\nu\bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{\alpha^2 \beta y_a^2}{\pi^2 q^2 \lambda_a} \\ & \quad \times G_2^a(y_a) \beta^2 \hat{k}_{ay} \sum_i G_i^s(x_1, x_2) P_{iy}, \end{aligned} \quad (153)$$

respectively.

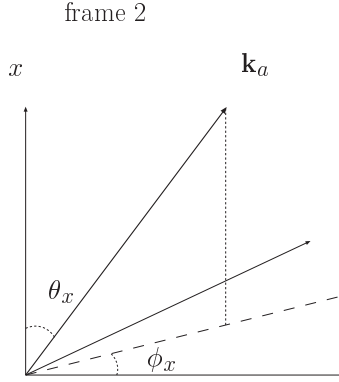


Figure 8: The polar coordinate of frame 2. The North pole is x direction.

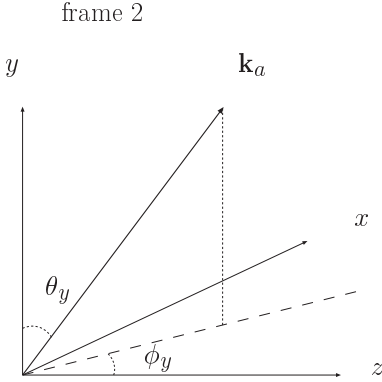


Figure 9: The polar coordinate of frame 2. The North pole is y direction.

From the above three equations, we pull out the quantities,

$$\begin{aligned} & \sum_i P_{ix} G_i^s(x_1, x_2) \\ & \sum_i P_{iy} G_i^s(x_1, x_2) \\ & \sum_i P_{iz} G_i^s(x_1, x_2). \end{aligned} \quad (154)$$

First, using these quantities, we analyze $G_1^s(x_1, x_2)$ and $G_2^s(x_1, x_2)$ contemporary. $G_{\hat{\mathbf{p}}_3}^s(x_1, x_2)$ which is the $\hat{\mathbf{p}}_3$ component of $\sum_i \mathbf{P}_i G_i^s(x_1, x_2)$ is

defined as

$$\begin{aligned}
G_{\hat{\mathbf{p}}_3}^s(x_1, x_2) &= \hat{\mathbf{p}}_3 \cdot \sum_i \mathbf{P}_i G_i^s(x_1, x_2) = \hat{\mathbf{p}}_3 \cdot \hat{\mathbf{p}}_1 G_1^s(x_1, x_2) + \hat{\mathbf{p}}_3 \cdot \hat{\mathbf{p}}_2 G_2^s(x_1, x_2) \\
&= \hat{p}_{3x} \sum_i P_{ix} G_i^s(x_1, x_2) + \hat{p}_{3y} \sum_i P_{iy} G_i^s(x_1, x_2) + \hat{p}_{3z} \sum_i P_{iz} G_i^s(x_1, x_2),
\end{aligned} \tag{155}$$

where

$$\begin{aligned}
\hat{\mathbf{p}}_3 \cdot \hat{\mathbf{p}}_1 &= 1 - 2 \frac{1 - x_2}{x_1 x_3} \\
\hat{\mathbf{p}}_3 \cdot \hat{\mathbf{p}}_2 &= 1 - 2 \frac{1 - x_1}{x_2 x_3}.
\end{aligned} \tag{156}$$

Integrating over x_2 and defining the parameters

$$\begin{aligned}
a_- &= \left(\frac{|g_1|^2}{16} + |g_3|^2 \right) - \left(\frac{|g_2|^2}{16} + |g_4|^2 \right) \\
b_- &= |g_5|^2 - |g_6|^2 \\
c_- &= |eA_R|^2 - |eA_L|^2 \\
d_- &= -(Re[g_3 eA_L^*] - Re[g_4 eA_R^*]) \\
e_- &= -(Re[g_6 eA_R^*] - Re[g_5 eA_L^*]),
\end{aligned} \tag{157}$$

$G_{\hat{\mathbf{p}}_3}^s(x_1, x_2)$ becomes

$$\begin{aligned}
G_{\hat{\mathbf{p}}_3}^s(x_1) &= \int_{1-x_1}^{x_1} dx_2 G_{\hat{\mathbf{p}}_3}^s(x_1, x_2) \\
&= \frac{1}{6} \left[4(2x_1 - 1) \left\{ a_-(2x_1 - 1)(5 - 4x_1) + 12d_-(2x_1 - 1) + 3e_-(3 - 2x_1) \right\} \right. \\
&\quad + 8c_- \left\{ 2(2x_1 - 1) \frac{(x_1^2 - 13x_1 + 13)}{(1 - x_1)} \right. \\
&\quad \left. \left. - 3(2x_1^2 - 2x_1 + 1) \log \left[\frac{1 - x_1}{x_1} \right] + 24 \log[2(1 - x_1)] \right\} \right. \\
&\quad \left. + b_- \left\{ (8x_1^2 - 32x_1 + 23)(1 - 2x_1) - 24(1 - x_1)^2 \log[2(1 - x_1)] \right\} \right].
\end{aligned} \tag{158}$$

To determine the value of coefficient c_- , we define the function

$$F_5(x_1) = \frac{3(1 - x_1)}{8} G_{\hat{\mathbf{p}}_3}^s(x_1). \tag{159}$$

The value of F_5 where $x_1 = 1$ is

$$F_5(x_1)|_{x_1=1} = c_-. \tag{160}$$

So we can determine the value of c_- .

Then, to determine the values of a_- , b_- , d_- and e_- , we define another function subtracting the term of coefficient c_- ,

$$F_6(x_1) = 6 \{G_{\mathbf{p}_3}^s(x_1) - (c_- \text{ term})\}, \quad (161)$$

where

$$\begin{aligned} (c_- \text{ term}) = 8c_- \left\{ 2(2x_1 - 1) \frac{(x_1^2 - 13x_1 + 13)}{(1 - x_1)} \right. \\ \left. - 3(2x_1^2 - 2x_1 + 1) \log\left[\frac{1 - x_1}{x_1}\right] + 24 \log[2(1 - x_1)] \right\}. \end{aligned} \quad (162)$$

We'll find that the 4 aspects of this function lead to determination of parameters a_- , b_- , d_- and e_- . The value of F_6 where $x_1 = 1$ is

$$F_{6a} = F_6(x_1)|_{x_1=1} = 4a_- + b_- + 48d_- + 12e_-. \quad (163)$$

The gradient of F_6 where $x_1 = 1$ leads

$$F_{6b} = \frac{1}{6} \frac{d}{dx_1} F_6(x_1) \Big|_{x_1=1} = 3b_- + 32d_-. \quad (164)$$

The gradient of F_6 where $x_1 = 1/2$ leads

$$F_{6c} = \frac{1}{6} \frac{d}{dx_1} F_6(x_1) \Big|_{x_1=\frac{1}{2}} = -b_- + 8e_-. \quad (165)$$

The integration value of F_6 leads

$$F_{6d} = 6 \int_{\frac{1}{2}}^1 dx_1 F_6(x_1) = 6a_- - b_- + 48d_- + 24e_-. \quad (166)$$

From previous four equations, we can determine the parameters a_- , b_- , d_- and e_- :

$$a_- = \frac{1}{8}(-10F_{6a} + 3F_{6b} - 9F_{6c} + 8F_{6d}), \quad (167)$$

$$b_- = \frac{1}{2}(-6F_{6a} + 3F_{6b} - 3F_{6c} + 4F_{6d}), \quad (168)$$

$$d_- = \frac{1}{64}(18F_{6a} - 7F_{6b} + 9F_{6c} - 12F_{6d}), \quad (169)$$

$$e_- = \frac{1}{16}(-6F_{6a} + 3F_{6b} - F_{6c} + 4F_{6d}). \quad (170)$$

Next, we analyze $G_3^s(x_1, x_2)$ using the relation

$$G_3^s(x_1, x_2) = \frac{(\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2)}{|\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2|^2} \cdot \sum_i \mathbf{P}_i G_1^s(x_1, x_2). \quad (171)$$

Integrating it over x_2 , we define the quantity $G_3^s(x_1)$ as

$$\begin{aligned}
G_3^s(x_1) &= \int_{1-x_1}^{x_1} dx_2 G_3^s(x_1, x_2) \\
&= \frac{x_1}{3(1-x_1)} \left[4f_+ \left\{ (2x_1-1)(2x_1^2-2x_1-1) + 3(x_1-1)x_1 \log\left[\frac{1-x_1}{x_1}\right] \right\} \right. \\
&\quad \left. + g_+ \left\{ (2x_1-1)(4x_1^2-10x_1+7) + 6(x_1-1)^2 \log\left[\frac{1-x_1}{x_1}\right] \right\} \right], \tag{172}
\end{aligned}$$

where

$$\begin{aligned}
f_+ &= -(Im[g_3 e A_L^*] + Im[g_4 e A_R^*]) \\
g_+ &= -(Im[g_6 e A_R^*] + Im[g_5 e A_L^*]). \tag{173}
\end{aligned}$$

Here, we define the function

$$F_7(x_1) = G_3^s(x_1) \frac{6(1-x_1)}{(1-2x_1)^2}. \tag{174}$$

Thus we determine f_+ and g_+ from

$$\begin{aligned}
F_7(x_1) \Big|_{x_1=1} &= -8f_+ + 2g_+ \\
F_7(x_1) \Big|_{x_1=\frac{1}{2}} &= 3g_+. \tag{175}
\end{aligned}$$

So we can determine the values of a_- , b_- , c_- , d_- , e_- , f_+ and g_+ separately. Adding this result to the result of the previous section i.e. a_+ , b_+ , c_+ , d_+ and e_+ , we can determine the values of $|g_1|^2/16 + |g_3|^2$, $|g_2|^2/16 + |g_4|^2$, $|g_5|^2$, $|g_6|^2$, $|A_R|^2$, $|A_L|^2$, $Re[g_3 A_L^*]$, $Re[g_4 A_R^*]$, $Re[g_5 A_L^*]$, $Re[g_6 A_R^*]$, $Im[g_3 A_L^*] + Im[g_4 A_R^*]$ and $Im[g_5 A_L^*] + Im[g_6 A_R^*]$, separately. Moreover, in some suitable cases, we can determine $|g_1|$ and $|g_3|$ (also $|g_2|$ and $|g_4|$), separately. To separate these are the main purpose of following subsection.

9 Physics Implication from $G_i^s(x_3)$ Distribution of μ_3

In sections 3 and 5, we studied that a_+ , b_+ , c_+ , d_+ and e_+ are determined by the energy distribution. In section 8, we also studied that a_- , b_- , c_- , d_- , e_- , f_+ and g_+ are determined by the energy distribution and the angular distribution.

Here, we study some special cases using these parameters. Using Eq. (108), each parameter defined in (157) and (173) are represented as

$$\begin{aligned}
a_- &= \frac{1}{16}(r_1^2 - r_2^2) + r_3^2 - r_4^2 \\
b_- &= r_5^2 - r_6^2 \\
c_- &= r_R^2 - r_L^2 \\
d_- &= -r_3 r_L \cos(\theta_3 - \theta_L) + r_4 r_R \cos(\theta_4 - \theta_R) \\
e_- &= -r_6 r_R \cos(\theta_6 - \theta_R) + r_5 r_L \cos(\theta_5 - \theta_L) \\
f_+ &= -r_3 r_L \sin(\theta_3 - \theta_L) - r_4 r_R \sin(\theta_4 - \theta_R) \\
g_+ &= -r_6 r_R \sin(\theta_6 - \theta_R) - r_5 r_L \sin(\theta_5 - \theta_L).
\end{aligned} \tag{176}$$

9.1 A sufficient condition for existence of scalar and/or pseudo scalar current

We make three types of relations which may reveal the existence of scalar and/or pseudo scalar current in this and following two subsections.

First, in this subsection, we give a sufficient condition for existence of g_1 and/or g_2 . Using the fact that

$$\left(r_3 r_4 \sin(\theta_3 - \theta_L) \sin(\theta_4 - \theta_R) + \sqrt{\frac{r_1^2}{16} + r_3^2} \sqrt{\frac{r_2^2}{16} + r_4^2} \right) \geq 0 \tag{177}$$

and the relation

$$\begin{aligned}
f_+^2 + \frac{d_+^2 + d_-^2}{2} - \frac{a_+ c_+ - a_- c_-}{2} \\
&= r_3^2 r_L^2 + r_4^2 r_R^2 + 2r_3 r_L r_4 r_R \sin(\theta_3 - \theta_L) \sin(\theta_4 - \theta_R) - \frac{a_+ c_+ - a_- c_-}{2} \\
&= -\frac{r_1^2}{16} \frac{c_+ - c_-}{2} - \frac{r_2^2}{16} \frac{c_+ + c_-}{2} + 2r_3 r_L r_4 r_R \sin(\theta_3 - \theta_L) \sin(\theta_4 - \theta_R),
\end{aligned} \tag{178}$$

we can make a relation

$$\begin{aligned}
&\frac{|g_1|^2 |eA_L|^2 + |g_2|^2 |eA_R|^2}{16} \\
&\geq \frac{r_1^2 r_L^2 + r_2^2 r_R^2}{16} - 2r_L r_R \left(r_3 r_4 \sin(\theta_3 - \theta_L) \sin(\theta_4 - \theta_R) + \sqrt{\frac{r_1^2}{16} + r_3^2} \sqrt{\frac{r_2^2}{16} + r_4^2} \right) \\
&= \frac{a_+ c_+ - a_- c_-}{2} - \frac{1}{2} \sqrt{(a_+^2 - a_-^2)(c_+^2 - c_-^2)} - f_+^2 - \frac{d_+^2 + d_-^2}{2}.
\end{aligned} \tag{179}$$

If

$$\frac{a_+ c_+ - a_- c_-}{2} - \frac{1}{2} \sqrt{(a_+^2 - a_-^2)(c_+^2 - c_-^2)} - f_+^2 - \frac{d_+^2 + d_-^2}{2} > 0, \tag{180}$$

then

$$|g_1|^2 |eA_L|^2 + |g_2|^2 |eA_R|^2 > 0. \quad (181)$$

So, in this case, we can get the result that g_1 and/or $g_2 \neq 0$.

9.2 What can we say about g_1 , g_2 and/or CP Violation

We now give relations which are convenience for determining the existence of scalar and/or pseudo scalar currents and/or CP violation of Lagrangian. One of them is

$$\frac{a_+ + a_-}{2} - \left(\frac{d_+ + d_-}{2}\right)^2 \frac{2}{c_+ - c_-} = r_3^2 \sin^2(\theta_3 - \theta_L) + \frac{r_1^2}{16} \geq 0. \quad (182)$$

If the left hand side becomes zero and $a_+ + a_- \neq 0$, then $\sin(\theta_3 - \theta_L) = g_1 = 0$ and $(a_+ + a_-)/2 = |g_3|^2$. On the other hand, if the left hand side is larger than zero, at least, $|g_3| \sin(\theta_3 - \theta_L) \neq 0$ or $g_1 \neq 0$. $|g_3| \sin(\theta_3 - \theta_L) \neq 0$ means CP violation. $g_1 \neq 0$ means that the scalar and/or pseudo scalar current exists. Similarly, we make the relation,

$$\frac{a_+ - a_-}{2} - \left(\frac{d_+ - d_-}{2}\right)^2 \frac{2}{c_+ + c_-} = r_4^2 \sin^2(\theta_4 - \theta_R) + \frac{r_2^2}{16} \geq 0. \quad (183)$$

If the left hand side becomes zero and $a_+ - a_- \neq 0$, then $\sin(\theta_4 - \theta_R) = g_2 = 0$ and $(a_+ - a_-)/2 = |g_4|^2$. On the other hand, if the left hand side is larger than zero, at least, $|g_4| \sin(\theta_4 - \theta_R) \neq 0$ or $g_2 \neq 0$. $|g_4| \sin(\theta_4 - \theta_R) \neq 0$ means CP violation. $g_2 \neq 0$ means the that scalar and/or pseudo scalar current exists.

9.3 Existence of g_1 and/or g_2 , or the Values of $Im[g_3 eA_L^*]$ and $Im[g_4 eA_R^*]$

Finally, we have two more relations. First one is

$$\begin{aligned} & \sqrt{\frac{a_+ + a_-}{2} \frac{c_+ - c_-}{2} - \left(\frac{d_+ + d_-}{2}\right)^2} + \sqrt{\frac{a_+ - a_-}{2} \frac{c_+ + c_-}{2} - \left(\frac{d_+ - d_-}{2}\right)^2} \\ &= r_L \sqrt{r_3^2 \sin^2(\theta_3 - \theta_L) + \frac{r_1^2}{16}} + r_R \sqrt{r_4^2 \sin^2(\theta_4 - \theta_R) + \frac{r_2^2}{16}} \\ &\geq r_3 r_L |\sin(\theta_3 - \theta_L)| + r_4 r_R |\sin(\theta_4 - \theta_R)| \\ &\geq |f_+|. \end{aligned} \quad (184)$$

It becomes an equality when

$$\begin{cases} g_1 = g_2 = 0 \\ Im[g_3 eA_L^*] Im[g_4 eA_R^*] \geq 0. \end{cases} \quad (185)$$

In that case, from the relation (184) and the conditions (185), $Im[g_3eA_L^*]$ and $Im[g_4eA_R^*]$ are determined as

$$\begin{aligned} Im[g_3eA_L^*] &= -\frac{f_+}{|f_+|} \sqrt{\frac{a_+ + a_-}{2} \frac{c_+ - c_-}{2} - \left(\frac{d_+ + d_-}{2}\right)^2} \\ Im[g_4eA_R^*] &= -\frac{f_+}{|f_+|} \sqrt{\frac{a_+ - a_-}{2} \frac{c_+ + c_-}{2} - \left(\frac{d_+ - d_-}{2}\right)^2}. \end{aligned} \quad (186)$$

If $f_+ = 0$, it means $Im[g_3eA_L^*] = Im[g_4eA_R^*] = 0$ since $Im[g_3eA_L^*]Im[g_4eA_R^*] \geq 0$.

We derive another relation from (178)

$$\begin{aligned} 2Im[g_3eA_L^*]Im[g_4eA_R^*] &= f_+^2 - \frac{a_+c_+ - a_-c_-}{2} + \frac{d_+^2 + d_-^2}{2} \\ &\quad + \frac{1}{16}(|g_1|^2|eA_L|^2 + |g_2|^2|eA_R|^2) \\ &\geq \frac{1}{2} \{2f_+^2 - (a_+c_+ - a_-c_-) + (d_+^2 + d_-^2)\}. \end{aligned} \quad (187)$$

When the equation of (184) is not an equality and $2f_+^2 - (a_+c_+ - a_-c_-) + (d_+^2 + d_-^2) \geq 0$, then we can derive, using (185), that $g_1 \neq 0$ and/or $g_2 \neq 0$. Furthermore, if $2f_+^2 - (a_+c_+ - a_-c_-) + (d_+^2 + d_-^2) > 0$, then $Im[g_3eA_L^*] \neq 0$ and $Im[g_4eA_R^*] \neq 0$ since $Im[g_3eA_L^*]Im[g_4eA_R^*] > 0$. In that case, CP symmetry is violated and the sign of $Im[g_3eA_L^*]$ and $Im[g_4eA_R^*]$ is the same as that of $-f_+$.

9.4 $Im[g_5eA_L^*]$ and $Im[g_6eA_R^*]$

Here, we give the method to determine $Im[g_5eA_L^*]$ and $Im[g_6eA_R^*]$.

Substituting the relations

$$\begin{aligned} Im[g_5eA_L^*]^2 &= |g_5|^2|eA_L|^2 - Re[g_5eA_L^*]^2 \\ Im[g_6eA_R^*]^2 &= |g_6|^2|eA_R|^2 - Re[g_6eA_R^*]^2 \end{aligned} \quad (188)$$

to

$$\begin{aligned} Im[g_5eA_L^*]^2 &= (g_+ + Im[g_6eA_R^*])^2 \\ &= Im[g_6eA_R^*]^2 + g_+^2 + 2g_+Im[g_6eA_R^*], \end{aligned} \quad (189)$$

the imaginary part of $g_6eA_R^*$ is represented only by the observables as

$$\begin{aligned} Im[g_6eA_R^*] &= \frac{-Im[g_6eA_R^*]^2 - g_+^2 + Im[g_5eA_L^*]^2}{2g_+} \\ &= -\frac{|g_6|^2|eA_R|^2 - Re[g_6eA_R^*]^2 + g_+^2 - |g_5|^2|eA_L|^2 + Re[g_5eA_L^*]^2}{2g_+} \\ &= -\frac{b_+c_- - b_-c_+ - 2e_+e_- + 2g_+^2}{4g_+}. \end{aligned} \quad (190)$$

So, if $g_+ \neq 0$, $Im[g_6 e A_R^*]$ can be determined independently. Similarly, $Im[g_5 e A_L^*]$ is represented as

$$\begin{aligned}
Im[g_5 e A_L^*] &= \frac{Im[g_5 e A_L^*]^2 + g_+^2 - Im[g_6 e A_R^*]^2}{2g_+} \\
&= \frac{|g_5|^2 |e A_L|^2 - Re[g_5 A_L^*]^2 + g_+^2 - |g_6|^2 |e A_R|^2 + Re[g_6 e A_R^*]^2}{2g_+} \\
&= -\frac{b_- c_+ - b_+ c_- + 2e_+ e_- + 2g_+^2}{4g_+},
\end{aligned} \tag{191}$$

if $g_+ \neq 0$.

Even if $g_+ = 0$,

$$\frac{c_+ + c_-}{2} \frac{b_+ - b_-}{2} - \left(\frac{e_+ + e_-}{2} \right)^2 = Im[g_6 e A_R^*]^2 \tag{192}$$

and

$$\frac{c_+ - c_-}{2} \frac{b_+ + b_-}{2} - \left(\frac{e_+ - e_-}{2} \right)^2 = Im[g_5 e A_L^*]^2. \tag{193}$$

So we can determine the absolute values of $Im[g_5 e A_L^*]$ and $Im[g_6 e A_R^*]$.

9.5 What can we say if one of $A_R, A_L, g_3, g_4 = 0$

If we restrict that $A_R = 0$ from other experiments or some specific models, then

$$\begin{aligned}
f_+ &= -r_3 r_L \sin(\theta_3 - \theta_L) \\
d_+ &= -r_3 r_L \cos(\theta_3 - \theta_L) \\
c_+ &= r_L^2.
\end{aligned} \tag{194}$$

So,

$$\begin{aligned}
\frac{d_+^2 + f_+^2}{c_+} &= r_3^2 = |g_3|^2 \\
\frac{a_+ + a_-}{2} - \frac{d_+^2 + f_+^2}{c_+} &= \frac{r_1^2}{16} = \frac{|g_1|^2}{16} \\
c_+ &= |e A_L|^2.
\end{aligned} \tag{195}$$

In this case, $|g_3|$ and $|g_1|$ are determined independently. Similarly, if $A_L = 0$,

$$\begin{aligned}
\frac{d_+^2 + f_+^2}{c_+} &= |g_4|^2 \\
\frac{a_+ - a_-}{2} - \frac{d_+^2 + f_+^2}{c_+} &= \frac{|g_2|^2}{16} \\
c_+ &= |e A_R|^2;
\end{aligned} \tag{196}$$

if $g_4 = 0$,

$$\begin{aligned} (d_+^2 + f_+^2) \frac{2}{c_+ - c_-} &= |g_3|^2 \\ \frac{a_+ + a_-}{2} - (d_+^2 + f_+^2) \frac{2}{c_+ - c_-} &= \frac{|g_1|^2}{16} \\ \frac{a_+ - a_-}{2} &= \frac{|g_2|^2}{16}, \end{aligned} \quad (197)$$

and if $g_3 = 0$,

$$\begin{aligned} (d_+^2 + f_+^2) \frac{2}{c_+ + c_-} &= |g_4|^2 \\ \frac{a_+ - a_-}{2} - (d_+^2 + f_+^2) \frac{2}{c_+ + c_-} &= \frac{|g_2|^2}{16} \\ \frac{a_+ + a_-}{2} &= \frac{|g_1|^2}{16}. \end{aligned} \quad (198)$$

9.6 CP violation

The master formula (9) contains a part,

$$\begin{aligned} (1 + \cos^2 \eta - \frac{\sin^2 \eta}{\gamma^2}) \hat{k}_{az} P_{3z} - \beta^2 \sin^2 \eta \hat{k}_{ay} P_{3y} \\ + (1 + \frac{1}{\gamma^2}) \sin^2 \eta \hat{k}_{ax} P_{3x} - \frac{\sin 2\eta}{\gamma} (\hat{k}_{ax} P_{3z} + \hat{k}_{az} P_{3x}). \end{aligned} \quad (199)$$

In this part, each term is proportional to

$$\hat{k}_{az}(\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_1)_z, \quad \hat{k}_{ay}(\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_1)_y, \quad \hat{k}_{ax}(\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_1)_x, \quad \hat{k}_{ax}(\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_1)_z + \hat{k}_{az}(\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_1)_x. \quad (200)$$

Supposing CPT theorem, time reversal is equivalent to CP transformation. In time reversal, momentum flips their directions. In fact,

$$\hat{\mathbf{k}}_a \rightarrow -\hat{\mathbf{k}}_a, \quad \hat{\mathbf{p}}_1 \rightarrow -\hat{\mathbf{p}}_1, \quad \hat{\mathbf{p}}_2 \rightarrow -\hat{\mathbf{p}}_2. \quad (201)$$

In this transformation, (200) flip their signs. This means (199) part in (9) flips its sign. This part proportions to g_+ or f_+ . These coefficients are the imaginary parts of the interactions. So, if they do not vanish, then CP symmetry is violated.

9.7 $A_L, A_R \gg$ others

If $\alpha Br(\tau \rightarrow \mu\gamma)$ is large compared to 4-fermi sector of $Br(\tau \rightarrow 3\mu)$, here α is the fine structure constant, it may be difficult to determine $|g_1|$ to $|g_6|$

directly from $\tau \rightarrow 3\mu$. It may be still possible to get a bound for these coupling constant from interference effects as follows.

$$|g_3|^2 = r_3^2 \geq r_3^2 \cos^2(\theta_3 - \theta_L) = \frac{(d_+ + d_-)^2}{2(c_+ - c_-)} \quad (202)$$

$$|g_4|^2 = r_4^2 \geq r_4^2 \cos^2(\theta_4 - \theta_R) = \frac{(d_+ - d_-)^2}{2(c_+ + c_-)} \quad (203)$$

$$|g_5|^2 = r_5^2 \geq r_5^2 \cos^2(\theta_5 - \theta_L) = \frac{(e_+ - e_-)^2}{2(c_+ - c_-)} \quad (204)$$

$$|g_6|^2 = r_6^2 \geq r_6^2 \cos^2(\theta_6 - \theta_R) = \frac{(e_+ + e_-)^2}{2(c_+ + c_-)}. \quad (205)$$

9.8 $A_L, A_R \ll$ others

If $\alpha Br(\tau \rightarrow \mu\gamma)$ is small compared to $Br(\tau \rightarrow 3\mu)$, we can still get some lower bounds from the amplitudes of $\tau \rightarrow \mu\gamma$ decay as follows.

$$|eA_L|^2 = r_L^2 \geq \frac{r_3^2 r_L^2 \cos^2(\theta_3 - \theta_L)}{\frac{r_1^2}{16} + r_3^2} = \frac{(d_+ + d_-)^2}{2(a_+ + a_-)} \quad (206)$$

$$|eA_L|^2 = r_L^2 \geq \frac{r_5^2 r_L^2 \cos^2(\theta_5 - \theta_L)}{r_5^2} = \frac{(e_+ - e_-)^2}{2(b_+ + b_-)} \quad (207)$$

$$|eA_R|^2 = r_R^2 \geq \frac{r_4^2 r_R^2 \cos^2(\theta_4 - \theta_R)}{\frac{r_2^2}{16} + r_4^2} = \frac{(d_+ - d_-)^2}{2(a_+ - a_-)} \quad (208)$$

$$|eA_R|^2 = r_R^2 \geq \frac{r_6^2 r_R^2 \cos^2(\theta_6 - \theta_R)}{r_6^2} = \frac{(e_+ + e_-)^2}{2(b_+ - b_-)} \quad (209)$$

10 Concluding Remarks

Neutrino oscillation suggests that, flavor quantum number isn't conserved not only in quark sector but also in neutrino sector. KM ansatz implies that if neutrino sector violate the flavor quantum number, charged lepton sector also do it. In the Standard Model, this violation is too small to determine in any designed futur experiments. However, in some new models, it is suggested that this violation is going to be determine. If $\tau \rightarrow 3\mu$ event is

detected, there are many models which are suitable to the first experimental result. However, at least, all models without one model is not true. So, if $\tau \rightarrow 3\mu$ event is detected, our analysis must be necessary to figure out if one model is allowed or forbidden.

We assumed only Lorentz and gauge invariance of the Lagrangian and locality of the action. So, if we cannot fit the data to the differential cross section, it means violation of Lorentz or gauge invariance.

From energy distributions $(|g_1|^2/16+|g_3|^2)+(|g_2|^2/16+|g_4|^2)$, $|g_5|^2+|g_6|^2$, $|A_R|^2+|A_L|^2$, $Re[g_4A_R^*]+Re[g_3A_L^*]$ and $Re[g_6A_R^*]+Re[g_5A_L^*]$ can be determined. Using the angular distribution of decaying products of τ^+ and τ^- , we can determine $(|g_1|^2/16+|g_3|^2)$, $(|g_2|^2/16+|g_4|^2)$, $|g_5|$, $|g_6|$, $|A_R|$, $|A_L|$, $Re[g_4A_R^*]$, $Re[g_3A_L^*]$, $Re[g_6A_R^*]$, $Re[g_5A_L^*]$, $Im[g_4A_R^*]+Im[g_3A_L^*]$ and $Im[g_6A_R^*]+Im[g_5A_L^*]$ independently. We can determine the argument of relative phases, $\arg[g_4A_R^*]$ and $\arg[g_3A_L^*]$, if $Im[g_4A_R^*]$ and $Im[g_3A_L^*]$ have same sign, and if there are no scalar and pseudo scalar interaction. Similarly, we can also determine the argument of relative phases, $\arg[g_6A_R^*]$ and $\arg[g_5A_L^*]$, if g_+ is nonzero.

Even if $\tau \rightarrow \mu\gamma$ process is suppressed by a factor of 100 or more than $\tau \rightarrow 3\mu$ process, we may still estimate the branching ratio of $\tau \rightarrow \mu\gamma$ from these interference:

$$\frac{\text{Br}(\tau \rightarrow 3\mu \text{ interference})}{\text{Br}(\tau \rightarrow \mu\gamma)} \sim \sqrt{\alpha} \left| \frac{g_3 + g_4 + g_5 + g_6}{A_R + A_L} \right| \gg 1, \quad (210)$$

where α is the fine structure constant. In concrete terms, for instance, even if $|eA_L|^2$ is too small to determine directly, we may be still able to get a lower bound $(d_+ + d_-)^2 / \{2(a_+ - a_-)\}$ as explained in subsection 9.8.

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A Fiertz Transformation

In Eq. (2), the terms

$$\begin{aligned}
& (\bar{\tau}_R \mu_L)(\bar{\mu}_L \mu_R), \\
& (\bar{\tau}_L \mu_R)(\bar{\mu}_R \mu_L), \\
& (\bar{\tau}_R \sigma_{\alpha\beta} \mu_L)(\bar{\mu}_R \sigma^{\alpha\beta} \mu_L), \\
& (\bar{\tau}_L \sigma_{\alpha\beta} \mu_R)(\bar{\mu}_L \sigma^{\alpha\beta} \mu_R), \\
& (\bar{\tau}_L \sigma_{\alpha\beta} \mu_R)(\bar{\mu}_R \sigma^{\alpha\beta} \mu_L), \\
& (\bar{\tau}_R \sigma_{\alpha\beta} \mu_L)(\bar{\mu}_L \sigma^{\alpha\beta} \mu_R),
\end{aligned} \tag{211}$$

which are naively assumed particularly don't appear. The reasons are as follows.

First, using the equations

$$\begin{aligned}
& (\bar{\psi}_{1R} \sigma^{\mu\nu} \psi_{2L})(\bar{\psi}_{3R} \sigma_{\mu\nu} \psi_{4L}) \\
& = -6(\bar{\psi}_{1R} \psi_{4L})(\bar{\psi}_{3R} \psi_{2L}) + \frac{1}{2}(\bar{\psi}_{1R} \sigma^{\mu\nu} \psi_{4L})(\bar{\psi}_{3R} \sigma_{\mu\nu} \psi_{2L})
\end{aligned} \tag{212}$$

and

$$\begin{aligned}
& (\bar{\psi}_{1R}\sigma^{\mu\nu}\psi_{4L})(\bar{\psi}_{3R}\sigma_{\mu\nu}\psi_{2L}) \\
& = -6(\bar{\psi}_{1R}\psi_{2L})(\bar{\psi}_{3R}\psi_{4L}) + \frac{1}{2}(\bar{\psi}_{1R}\sigma^{\mu\nu}\psi_{2L})(\bar{\psi}_{3R}\sigma_{\mu\nu}\psi_{4L}),
\end{aligned} \tag{213}$$

where the $\psi_{iL} = (1 - \gamma_5)\psi_i/2$, $\psi_{iR} = (1 + \gamma_5)\psi_i/2$, $\bar{\psi}_{iL} = \bar{\psi}_i(1 + \gamma_5)/2$, $\bar{\psi}_{iR} = \bar{\psi}_i(1 - \gamma_5)/2$ and ψ_i where $i = \{1, 2, 3, 4\}$ are the Dirac spinors. Then,

$$\begin{aligned}
& (\bar{\psi}_{1R}\sigma^{\mu\nu}\psi_{2L})(\bar{\psi}_{3R}\sigma_{\mu\nu}\psi_{4L}) - (\bar{\psi}_{1R}\sigma^{\mu\nu}\psi_{4L})(\bar{\psi}_{3R}\sigma_{\mu\nu}\psi_{2L}) \\
& = -4(\bar{\psi}_{1R}\psi_{4L})(\bar{\psi}_{3R}\psi_{2L}) + 4(\bar{\psi}_{1R}\psi_{2L})(\bar{\psi}_{3R}\psi_{4L})
\end{aligned} \tag{214}$$

and also exchanging L and R,

$$\begin{aligned}
& (\bar{\psi}_{1L}\sigma^{\mu\nu}\psi_{2R})(\bar{\psi}_{3L}\sigma_{\mu\nu}\psi_{4R}) - (\bar{\psi}_{1L}\sigma^{\mu\nu}\psi_{4R})(\bar{\psi}_{3L}\sigma_{\mu\nu}\psi_{2R}) \\
& = -4(\bar{\psi}_{1L}\psi_{4R})(\bar{\psi}_{3L}\psi_{2R}) + 4(\bar{\psi}_{1L}\psi_{2R})(\bar{\psi}_{3L}\psi_{4R}).
\end{aligned} \tag{215}$$

Next,

$$(\bar{\psi}_{1R}\sigma^{\mu\nu}\psi_{2L})(\bar{\psi}_{3L}\sigma_{\mu\nu}\psi_{4R}) = \sum_i C_i (\bar{\psi}_{1R}\Gamma_i\psi_{4R})(\bar{\psi}_{3L}\Gamma_i\psi_{2L}) = 0 \tag{216}$$

where $\Gamma_i = \{1, \gamma_5, \sigma^{\mu\nu}\}$ and C_i are some coefficients.

Finally,

$$(\bar{\psi}_{1L}\psi_{2R})(\bar{\psi}_{3R}\psi_{4L}) = -2(\bar{\psi}_{1L}\gamma^\mu\psi_{4R})(\bar{\psi}_{3L}\gamma_\mu\psi_{2R}). \tag{217}$$

Then,

$$\begin{aligned}
& (\bar{\psi}_{1L}\psi_{2R})(\bar{\psi}_{3R}\psi_{4L}) - (\bar{\psi}_{1L}\psi_{4R})(\bar{\psi}_{3R}\psi_{2L}) \\
& = 2(\bar{\psi}_{1L}\gamma^\mu\psi_{2R})(\bar{\psi}_{3L}\gamma_\mu\psi_{4R}) - 2(\bar{\psi}_{1L}\gamma^\mu\psi_{4R})(\bar{\psi}_{3L}\gamma_\mu\psi_{2R})
\end{aligned} \tag{218}$$

and also

$$\begin{aligned}
& (\bar{\psi}_{1R}\psi_{2L})(\bar{\psi}_{3L}\psi_{4R}) - (\bar{\psi}_{1R}\psi_{4L})(\bar{\psi}_{3L}\psi_{2R}) \\
& = 2(\bar{\psi}_{1R}\gamma^\mu\psi_{2L})(\bar{\psi}_{3R}\gamma_\mu\psi_{4L}) - 2(\bar{\psi}_{1R}\gamma^\mu\psi_{4L})(\bar{\psi}_{3R}\gamma_\mu\psi_{2L}).
\end{aligned} \tag{219}$$

So it is proved that Eq. (2) is the general form of 4-Fermi type interactions.

B Parts of Master Formula

B.1 production cross section of τ pair with the polarizations

Defining s^\pm as τ^\pm polarization vector in frame 3 and 2, respectively, the differential cross section for the process $e^+e^- \rightarrow \gamma^* \rightarrow \tau^+(s^+)\tau^-(s^-)$ in the

center of mass frame, frame 1 is [12]:

$$\begin{aligned}
& \frac{d\sigma(e^+e^- \rightarrow \tau^+(s^+)\tau^-(s^-))}{d\Omega} \\
&= \frac{\alpha^2\beta}{4q^2} \left[\left(1 + \cos^2\eta + \frac{\sin^2\eta}{\gamma^2}\right) + \left(1 + \cos^2\eta - \frac{\sin^2\eta}{\gamma^2}\right)s_z^+s_z^- - \beta^2\sin^2\eta s_y^+s_y^- \right. \\
&\quad \left. + \left(1 + \frac{1}{\gamma^2}\right)\sin^2\eta s_x^+s_x^- - \frac{\sin 2\eta}{\gamma}(s_z^+s_x^- + s_x^+s_z^-) \right], \tag{220}
\end{aligned}$$

where $\alpha \simeq 1/137$ is the fine structure constant, $\beta = |\mathbf{p}'_\tau|/E$, $\gamma = (1 - \beta^2)^{-1/2} = E/m_\tau$, E is the energy of e^+ or e^- in the initial state and $|\mathbf{p}'_\tau|$ is the absolute value of momentum of τ^+ . As described in Fig. 3, η is the angle between the momenta of e^+ in the initial state and τ^+ . $q^2 = (p'_{e^+} + p'_{e^-})^2$, where p'_{e^+} and p'_{e^-} are the momenta of e^+ and e^- in the initial state, respectively. Ω is the solid angle for the τ^+ momentum. We note that the quantities, E , $|\mathbf{p}'_\tau|$, p'_{e^\pm} and Ω are defined in frame 1.

B.2 differential branching ratio for τ^- decay

The differential Branching ratio for the process $\tau^- \rightarrow \nu_\tau + a + \text{anything}$ in the rest frame of τ^- is [13];

$$\begin{aligned}
& \frac{dBr(\tau^-(s^-) \rightarrow \nu_\tau + a + \text{anything})}{d^3k_a} \\
&= Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \frac{2}{\pi m_\tau^3 \lambda_a} \left[G_1^a(y_a) - \mathbf{s}^- \cdot \hat{\mathbf{k}}_a G_2^a(y_a) \right], \tag{221}
\end{aligned}$$

where $G_1^a(y_a)$ and $G_2^a(y_a)$ are the functions of y_a defined in each a . These are written in the table 1 of Ref. [13]. Here,

$$\lambda_a = \int dy_a y_a^2 G_1^a(y_a), \tag{222}$$

$$y_a = \frac{2E_a}{m_\tau}, \tag{223}$$

E_a is the energy of a , $\hat{\mathbf{k}}_a = \mathbf{k}_a/|\mathbf{k}_a|$ and k_a is the momentum of the particle a .

We note here that physical vector quantities which we treat in this process are only \mathbf{s}^- and $\hat{\mathbf{k}}_a$. The only scalar made by these vector quantities is $\mathbf{s}^- \cdot \hat{\mathbf{k}}_a$. So, we can explain the differential branching ratio, Eq. (221) by only two terms which are proportional to $G_1^a(y_a)$ and $\mathbf{s}^- \cdot \hat{\mathbf{k}}_a G_2^a(y_a)$, respectively.

B.3 narrow width approximation

The narrow width approximation is

$$\frac{1}{|p^2 - (m - i\Gamma/2)|^2} \simeq \frac{\pi}{m\Gamma} \delta(p^2 - m^2), \quad \text{where} \quad \frac{\Gamma}{m} \ll 1. \quad (224)$$

B.4 the total Branching ratio

In Eq. (11), if $\sqrt{q^2} = m_{\Upsilon(4s)}$ which is the Upsilon 4S mass, the differential cross section in the center of mass frame, frame 1 is

$$\begin{aligned} & \frac{d\sigma}{dx_1 dx_2} \\ &= G_0(x_1, x_2) \frac{64\pi\alpha^2}{m_{\Upsilon(4s)}^2} \left(1 + \frac{2m_\tau^2}{m_{\Upsilon(4s)}^2} \right) \sqrt{1 - \frac{4m_\tau^2}{m_{\Upsilon(4s)}^2}} \\ & \times Br(\tau \rightarrow \mu\nu\bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}). \end{aligned} \quad (225)$$

Then, the total cross section is

$$\begin{aligned} \sigma &= \int_0^1 \int_{1-\frac{x_3}{2}}^{1-(\frac{4\delta}{3})^2} \frac{d\sigma}{dx_1 dx_3} dx_1 dx_3 = \frac{64\pi\alpha^2}{3m_{\Upsilon(4s)}^2} \left(1 + \frac{2m_\tau^2}{m_{\Upsilon(4s)}^2} \right) \sqrt{1 - \frac{4m_\tau^2}{m_{\Upsilon(4s)}^2}} \\ & \times Br(\tau \rightarrow \mu\nu\bar{\nu}) Br(\tau^- \rightarrow \nu_\tau + a + \text{anything}) \\ & \times \left[\frac{1}{2}a_+ + \frac{1}{4}b_+ + 4d_+ + 2e_+ + \frac{2}{3} \left(24 \log \left[\frac{3}{4\delta} \right] - 13 \right) c_+ \right] \end{aligned} \quad (226)$$

where

$$\frac{2}{3} \left(24 \log \left[\frac{3}{4\delta} \right] - 13 \right) \simeq 20.8$$

since $\delta = 2m_\mu/m_\tau$. Here, we derive the total cross section in center of mass frame of initial electron and positron since the total cross section is Lorentz invariant under the boost for the beam direction. So, this expression is able to apply to the B Factory experiment.

C G's

Using the width $\Gamma(\tau \rightarrow \mu\nu\bar{\nu}) = m_\tau^5 G_F^2 / (192\pi^3)$, the differential branching ratio concerning the polarization of τ is

$$\frac{dBr(\tau^+(s^+) \rightarrow \mu_1\mu_2\mu_3)}{dx_1 dx_2 d\Omega_\tau d\psi} = \frac{3}{2\pi^2} Br(\tau \rightarrow \mu\nu\bar{\nu}) \left[G_0(x_1, x_2) + \mathbf{s}^+ \cdot \mathbf{P}_i G_i^s(x_1, x_2) \right]. \quad (227)$$

$G_0(x_1, x_2)$ and $G_i^s(x_1, x_2)$ are as follows.

C.1 $G_0(x_1, x_2)$

$$\begin{aligned}
G_0(x_1, x_2) = & 4a_+(2 - x_1 - x_2)(x_1 + x_2 - 1) + b_+((1 - x_1)x_1 + (1 - x_2)x_2) \\
& + 4c_+ \frac{(1 - x_1)(x_1^2 + (1 - x_1)^2) + (1 - x_2)(x_2^2 + (1 - x_2)^2)}{(1 - x_1)(1 - x_2)} \\
& + 16d_+(x_1 + x_2 - 1) + 4e_+(2 - x_1 - x_2)
\end{aligned} \tag{228}$$

C.2 $G_i^s(x_1, x_2)$

$$\begin{aligned}
G_1^s(x_1, x_2) = & -4a_-x_1(x_1 + x_2 - 1) + b_-x_1(1 - x_1) \\
& + 4c_- \frac{x_1(2x_1(1 - x_1) + 2(x_1 + x_2 - 1) - 1)}{(1 - x_1)(1 - x_2)} \\
& - 4d_- \frac{x_1(x_1 + x_2 - 1)(-x_2^2 + x_1(x_2 - 2) + 2)}{(1 - x_1)(1 - x_2)} \\
& - 2e_- \frac{x_1((2 - x_2)(1 - x_1)^2 + x_2(1 - x_2)^2)}{(1 - x_1)(1 - x_2)}
\end{aligned} \tag{229}$$

$$G_2^s(x_1, x_2) = G_1^s(x_2, x_1) \tag{230}$$

$$G_3^s(x_1, x_2) = \left[-2f_+(x_1 + x_2 - 1) - g_+(x_1 + x_2 - 2) \right] \frac{2x_1(x_1 - x_2)x_2}{(x_1 - 1)(x_2 - 1)} \tag{231}$$

D derivations of relations, $a_+c_+ - d_+^2$ and $b_+c_+ - e_+^2$

Here, we introduce a convenient formulae $a_+c_+ - d_+^2$ and $b_+c_+ - e_+^2$ to use in sections 4, 6, 7 and 9. Using this formula, we can determine more about current structure.

Using the relations

$$\begin{aligned}
& a_+c_+ - |g_3eA_L^* + g_4eA_R^*|^2 \\
& = \frac{1}{16}(r_1^2 + r_2^2)(r_L^2 + r_R^2) + (r_3^2 + r_4^2)(r_L^2 + r_R^2) \\
& \quad - r_3^2r_L^2 - r_4^2r_R^2 - 2r_3r_Lr_4r_R \cos(\theta_3 - \theta_L - \theta_4 + \theta_R) \\
& = r_3^2r_R^2 - 2r_3r_Rr_4r_L \cos(\theta_3 + \theta_R - \theta_4 - \theta_L) + r_4^2r_L^2 + \frac{1}{16}(r_1^2 + r_2^2)(r_L^2 + r_R^2) \\
& = |r_3e^{i\theta_3}r_Re^{i\theta_R} - r_4e^{i\theta_4}r_Le^{i\theta_L}|^2 + \frac{1}{16}(r_1^2 + r_2^2)(r_L^2 + r_R^2) \\
& = |g_3eA_R - g_4eA_L|^2 + \frac{1}{16}(r_1^2 + r_2^2)(r_L^2 + r_R^2)
\end{aligned} \tag{232}$$

and

$$\begin{aligned}
& |g_3 e A_L^* + g_4 e A_R^*|^2 - d_+^2 \\
&= r_3^2 r_L^2 + r_4^2 r_R^2 + 2r_3 r_L r_4 r_R \cos(\theta_3 - \theta_L - \theta_4 + \theta_R) \\
&\quad - r_3^2 r_L^2 \cos^2(\theta_3 - \theta_L) - 2r_3 r_L r_4 r_R \cos(\theta_3 - \theta_L) \cos(\theta_4 - \theta_R) - r_4^2 r_R^2 \cos^2(\theta_4 - \theta_R) \\
&= r_3^2 r_L^2 \sin^2(\theta_3 - \theta_L) + 2r_3 r_L r_4 r_R \sin(\theta_3 - \theta_L) \sin(\theta_4 - \theta_R) + r_4^2 r_R^2 \sin^2(\theta_4 - \theta_R) \\
&= \{r_3 r_L \sin(\theta_3 - \theta_L) + r_4 r_R \sin(\theta_4 - \theta_R)\}^2 \\
&= (Im[g_3 e A_L^*] + Im[g_4 e A_R^*])^2,
\end{aligned} \tag{233}$$

we give a useful relation

$$\begin{aligned}
& a_+ c_+ - d_+^2 \\
&= \frac{1}{16} (r_1^2 + r_2^2)(r_L^2 + r_R^2) + |r_3 e^{i\theta_3} r_R e^{i\theta_R} - r_4 e^{i\theta_4} r_L e^{i\theta_L}|^2 \\
&\quad + \{r_3 r_L \sin(\theta_3 - \theta_L) + r_4 r_R \sin(\theta_4 - \theta_R)\}^2 \\
&= \frac{1}{16} (|g_1|^2 + |g_2|^2)(|e A_L|^2 + |e A_R|^2) + |g_3 e A_R - g_4 e A_L|^2 \\
&\quad + (Im[g_3 e A_L^*] + Im[g_4 e A_R^*])^2 \geq 0.
\end{aligned} \tag{234}$$

In this formula, there are three terms. Each of them has zero or positive value. First term $(|g_1|^2 + |g_2|^2)(|e A_L|^2 + |e A_R|^2)/16$ has the information about scalar and pseudo scalar currents. Second term $|g_3 e A_R - g_4 e A_L|^2$ has information about parity symmetry. Third term $(Im[g_3 e A_L^*] + Im[g_4 e A_R^*])^2$ has information about CP symmetry.

If $a_+ = 0$, then $r_1 = r_2 = r_3 = r_4 = 0$ and also $d_+ = 0$. Similarly, if $c_+ = 0$, then $r_R = r_L = 0$ and also $d_+ = 0$. In these cases, we can't use interference effect. So, we consider only the case

$$\begin{aligned}
& a_+ \neq 0 \\
& c_+ \neq 0
\end{aligned} \tag{235}$$

when we use this formula.

Similarly,

$$\begin{aligned}
& b_+ c_+ - e_+^2 \\
&= |r_5 e^{i\theta_5} r_R e^{i\theta_R} - r_6 e^{i\theta_6} r_L e^{i\theta_L}|^2 + \{r_5 r_L \sin(\theta_5 - \theta_L) + r_6 r_R \sin(\theta_6 - \theta_R)\}^2 \\
&= |g_5 e A_R - g_6 e A_L|^2 + (Im[g_5 e A_L^*] + Im[g_6 e A_R^*])^2 \geq 0.
\end{aligned} \tag{236}$$

In this formula, there are two terms. Each of them has zero or positive value. First term $|g_5 e A_R - g_6 e A_L|^2$ has information about parity symmetry. Second term $(Im[g_5 e A_L^*] + Im[g_6 e A_R^*])^2$ has information about CP symmetry.

If $b_+ = 0$, then $r_5 = r_6 = 0$ and also $e_+ = 0$. Similarly, if $c_+ = 0$, then $r_R = r_L = 0$ and also $e_+ = 0$. In these cases, we can't use interference effect. So, we consider only the case

$$\begin{aligned} b_+ &\neq 0 \\ c_+ &\neq 0 \end{aligned} \quad (237)$$

when we use this formula.

E C, P, T and CP Transformation

We define charge (C), parity (P), time reversal (T) and CP transformations.

In Lagrangian, C transformation is

$$\begin{aligned} g_1 &\longleftrightarrow g_2^* & g_2 &\longleftrightarrow g_1^* \\ g_3 &\longleftrightarrow g_4^* & g_4 &\longleftrightarrow g_3^* \\ g_5 &\longleftrightarrow g_6^* & g_6 &\longleftrightarrow g_5^* \\ A_R &\longleftrightarrow A_L^* & A_L &\longleftrightarrow A_R^*, \end{aligned} \quad (238)$$

P transformation is

$$\begin{aligned} g_1 &\longleftrightarrow g_2 & g_1^* &\longleftrightarrow g_2^* \\ g_3 &\longleftrightarrow g_4 & g_3^* &\longleftrightarrow g_4^* \\ g_5 &\longleftrightarrow g_6 & g_5^* &\longleftrightarrow g_6^* \\ A_R &\longleftrightarrow A_L & A_R^* &\longleftrightarrow A_L^* \end{aligned} \quad (239)$$

and T and CP transformations are

$$\begin{aligned} g_1 &\longleftrightarrow g_1^* & g_2 &\longleftrightarrow g_2^* \\ g_3 &\longleftrightarrow g_3^* & g_4 &\longleftrightarrow g_4^* \\ g_5 &\longleftrightarrow g_5^* & g_6 &\longleftrightarrow g_6^* \\ A_R &\longleftrightarrow A_R^* & A_L &\longleftrightarrow A_L^*. \end{aligned} \quad (240)$$

F Observables in e^+e^- Center of Mass Frame

The relation

$$\frac{d\sigma}{d\alpha_1 d\alpha_2 \cdots d\alpha_n} = \int \frac{d\sigma}{d\beta_1 d\beta_2 \cdots d\beta_m} \prod_{i=1}^n \delta(\alpha_i - \alpha_i(\beta_1, \beta_2, \cdots, \beta_m)) d\beta_1 d\beta_2 \cdots d\beta_m \quad (241)$$

is useful for converting the variables from $\beta_1, \beta_2, \cdots, \beta_m$ to $\alpha_1, \alpha_2, \cdots, \alpha_n$ [12].

To write it briefly, we define $s\eta$, $c\eta$, $s\phi$, $c\phi$, $s\theta$, $c\theta$, $s\psi$, $c\psi$, $s\xi$, $c\xi$, $s\chi$ and $c\chi$ as $\sin \eta$, $\cos \eta$, $\sin \phi$, $\cos \phi$, $\sin \theta$, $\cos \theta$, $\sin \psi$, $\cos \psi$, $\sin \xi$, $\cos \xi$, $\sin \chi$ and $\cos \chi$, respectively. Also we define the quantities p'_1, p'_2, p'_3, k'_a as the quantities p_1, p_2, p_3, k_a in frame 1. First,

$$\begin{aligned}
p'_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\eta & 0 & s\eta \\ 0 & 0 & 1 & 0 \\ 0 & -s\eta & 0 & c\eta \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\phi & -s\phi & 0 \\ 0 & s\phi & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & 0 & s\theta \\ 0 & 0 & 1 & 0 \\ 0 & -s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} E_3 \\ 0 \\ 0 \\ E_3 \end{pmatrix} \\
&= E_3 \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ s\eta\gamma\beta & c\eta & 0 & s\eta\gamma \\ 0 & 0 & 1 & 0 \\ c\eta\gamma\beta & -s\eta & 0 & c\eta\gamma \end{pmatrix} \begin{pmatrix} 1 \\ s\theta c\phi \\ s\theta s\phi \\ c\theta \end{pmatrix} \\
&= E_3 \begin{pmatrix} \gamma(1 + \beta c\theta) \\ c\eta s\theta c\phi + s\eta\gamma(\beta + c\theta) \\ s\theta s\phi \\ -s\eta s\theta c\phi + c\eta\gamma(\beta + c\theta) \end{pmatrix}. \tag{242}
\end{aligned}$$

Here, θ and ϕ which are defined in Fig. 4 represent the direction of \mathbf{p}_3 in frame 3. η represents the direction of τ^+ in frame 1.

Next,

$$\begin{aligned}
p'_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\eta & 0 & s\eta \\ 0 & 0 & 1 & 0 \\ 0 & -s\eta & 0 & c\eta \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\phi & -s\phi & 0 \\ 0 & s\phi & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & 0 & s\theta \\ 0 & 0 & 1 & 0 \\ 0 & -s\theta & 0 & c\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\psi & -s\psi & 0 \\ 0 & s\psi & c\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\xi & 0 & s\xi \\ 0 & 0 & 1 & 0 \\ 0 & -s\xi & 0 & c\xi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} E_1 \\
&= \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ s\eta\gamma\beta & c\eta & 0 & s\eta\gamma \\ 0 & 0 & 1 & 0 \\ c\eta\gamma\beta & -s\eta & 0 & c\eta\gamma \end{pmatrix} \begin{pmatrix} 1 \\ c\phi(c\theta c\psi s\xi + s\theta c\xi) - s\phi(s\psi s\xi) \\ s\phi(c\theta c\psi s\xi + s\theta c\xi) + c\phi(s\psi s\xi) \\ -s\theta c\psi s\xi + c\theta c\xi \end{pmatrix} E_1, \tag{243}
\end{aligned}$$

where ψ is the angle between \mathbf{p}_2 - \mathbf{p}_3 plane and \mathbf{p}_3 - z plane as represented in

Fig. 5. Similarly,

$$p'_2 = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ s\eta\gamma\beta & c\eta & 0 & s\eta\gamma \\ 0 & 0 & 1 & 0 \\ c\eta\gamma\beta & -s\eta & 0 & c\eta\gamma \end{pmatrix} \begin{pmatrix} 1 \\ c\phi(c\theta c\psi s\chi + s\theta c\chi) - s\phi(s\psi s\chi) \\ s\phi(c\theta c\psi s\chi + s\theta c\chi) + c\phi(s\psi s\chi) \\ -s\theta c\psi s\chi + c\theta c\chi \end{pmatrix} E_2. \quad (244)$$

Here, as Eq. (156),

$$\begin{aligned} \cos \xi &= 1 - 2 \frac{1 - x_2}{x_1 x_3} = \hat{\mathbf{p}}_3 \cdot \hat{\mathbf{p}}_1 \\ \sin \xi &= \sqrt{1 - \cos^2 \xi} \end{aligned} \quad (245)$$

and

$$\begin{aligned} \cos \chi &= 1 - 2 \frac{1 - x_1}{x_2 x_3} = \hat{\mathbf{p}}_3 \cdot \hat{\mathbf{p}}_2 \\ \sin \chi &= -\sqrt{1 - \cos^2 \chi}, \end{aligned} \quad (246)$$

where $\sin \chi \leq 0$ since $\pi \leq \chi \leq 2\pi$.

Finally, we use the 3 ways to explain k'_a for the benefit of the simplicity of the phase space integral. The coordinates are as Figs. 7, 8 and 9, respectively.

$$\begin{aligned} k'_a &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c\eta & 0 & s\eta \\ 0 & 0 & 1 & 0 \\ 0 & -s\eta & 0 & c\eta \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_a \\ |\mathbf{k}_a| \sin \theta_{az} \cos \phi_{az} \\ |\mathbf{k}_a| \sin \theta_{az} \sin \phi_{az} \\ |\mathbf{k}_a| \cos \theta_{az} \end{pmatrix} \\ &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ -s\eta\gamma\beta & c\eta & 0 & s\eta\gamma \\ 0 & 0 & 1 & 0 \\ -c\eta\gamma\beta & -s\eta & 0 & c\eta\gamma \end{pmatrix} \begin{pmatrix} E_a \\ |\mathbf{k}_a| \sin \theta_{az} \cos \phi_{az} \\ |\mathbf{k}_a| \sin \theta_{az} \sin \phi_{az} \\ |\mathbf{k}_a| \cos \theta_{az} \end{pmatrix}. \end{aligned} \quad (247)$$

$$k'_a = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ -s\eta\gamma\beta & c\eta & 0 & s\eta\gamma \\ 0 & 0 & 1 & 0 \\ -c\eta\gamma\beta & -s\eta & 0 & c\eta\gamma \end{pmatrix} \begin{pmatrix} E_a \\ |\mathbf{k}_a| \cos \theta_{ax} \\ |\mathbf{k}_a| \sin \theta_{ax} \cos \phi_{ax} \\ |\mathbf{k}_a| \sin \theta_{ax} \sin \phi_{ax} \end{pmatrix}. \quad (248)$$

$$k'_a = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ -s\eta\gamma\beta & c\eta & 0 & s\eta\gamma \\ 0 & 0 & 1 & 0 \\ -c\eta\gamma\beta & -s\eta & 0 & c\eta\gamma \end{pmatrix} \begin{pmatrix} E_a \\ |\mathbf{k}_a| \sin \theta_{ay} \sin \phi_{ay} \\ |\mathbf{k}_a| \cos \theta_{ay} \\ |\mathbf{k}_a| \sin \theta_{ay} \cos \phi_{ay} \end{pmatrix}. \quad (249)$$